Theorem & Problems by Mathematical Induction

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INTRODUCTION

The mathematical induction is a peculiar type of method to prove the theorem and problems by using one to one correspondence with the set of positive integer and the variable or exponent involves in the statement.

Let a conjecture of the form:

$$\forall_n \ a_n = (n \in Z^+)$$

[or, for example, more generally :

$\forall_n \ a_n \ (n \in Z^+ \ and \ n > r)$, where r is a positive integer] In describing , proof by mathematical induction, we shall assume initially than n ranges over the entire set Z^+.

The method consists of two parts :

(i) First, we prove that a_n is true for n = 1

(ii) Secondly, we prove that if a_n is assumed to be TRUE for n equal to some arbitrary $k \in Z^+$, then it follows that a_n is TRUE also for n = k + 1.

Having proved (1) and (2) we then argue as follows :

Since a_n is TRUE for n = 1

It must also be TRUE for n = 1 + 1 = 2

and for n = 2 + 1 = 3

and for n = 3 + 1 = 4

and so on.

Hence a_n is TRUE for all $n \in Z^+$

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Example – 1

To prove $\forall_n\ D^n(x \mapsto xe^x) = x \mapsto (x+n)e^x\ (n \in Z^+)$ where D is the differentiation operator, and the functions both have domain R.

PROOF BY MATHEMATICAL INDUCTION

FIRST STEP

Consider the case n = 1

$$D^n(x \mapsto xe^x) = x \mapsto xe^x + e^x \text{ (product rule for differentiation)}$$

$$= x \mapsto (x+1)e^x \text{ (distributivity)}$$

SECOND STEP

Assume $D^k(x \mapsto xe^x) = x \mapsto (x+k)e^x\ (k \in Z^+)$ (hypothesis)

$$D^{k+1}(x \mapsto xe^x) = D[D^k(x \mapsto xe^x)] \text{ (definition of diff. operator)}$$

$$= D[x \mapsto (x+k)e^x] \text{ (hypothesis)}$$

$$= x \mapsto (x+k)e^x + e^x \text{ (product rule for diff)}$$

$$= x \mapsto [x+(k+1)e^x] \text{ (dist. and associativity)}$$

Since the conjecture is proved to be TRUE for n = 1, and also for n = k+1 whenever it is TRUE for $n = k(k \in Z^+)$, it follows that it is TRUE for n = 2, 3, 4, ... and so for $n \in Z^+$. The conjecture is therefore proved by mathematical induction to be TRUE, and can be accepted as a theorem of the calculus.

Example – 2

TO PROVE : The sum, S_n, of the interior angles of a convex polygon of n sides is $180(n-2)^0$.

PROOF BY MATHEMATICAL INDUCTION :

FIRST STEP

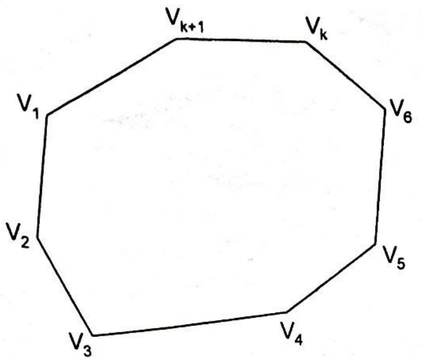
Consider the case n = 3

$$180(3-2)^0 = 180^0$$

Which is the sum of the interior angles of a triangle (a theorem of Euclidean geometry); so $180(3-2)^0 = S_3$.

SECOND STEP

Consider a convex polygon of k+1 sides, as show:



If we connect vertex V_1 to V_k (as shown) then the polygon is divided into a triangle and a convex polygon of k sides.

Assume

$$S_k = 180(k-2)^0 \qquad (k \in Z^+ \text{ and } k > 2)$$

$$\begin{aligned} S_{k+1} &= S_k + S_3 \\ &= S_k + 180^0 \\ &= 180(k-2)^0 + 180^0 \\ &= 180(k-1)^0 \\ &= 180[(k+1)-2]^0 \end{aligned}$$

The proof is now completed as the two steps have been-proved.

CONCLUSION :

It is important to note that the whole basis of proof by mathematical induction depends upon certain properties of the positive integers. Sometimes for the proof of few theorem and problems have no alternatives than mathematical induction.

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