# Riemann hypothesis and Zeta- Function

**Mani Raj Bhattrai***

## Introduction

In mathematics, the Riemann hypothesis, proposed by Bernhard Riemann (1859), is a conjecture about the distribution of the zeros of the Riemann zeta-function which states that all non-trivial zeros of the Riemann zeta function have real part 1/2. The name is also used for some closely related analogues, such as the Riemann hypothesis for curves over finite fields.

## The Riemann Zeta Function

Some numbers have the special property that they cannot be expressed as the product of two smaller numbers, e.g., 2, 3, 5, 7, etc. Such numbers are called *prime* numbers, and they play an important role, both in pure mathematics and its applications. The distribution of such prime numbers among all natural numbers does not follow any regular pattern, however the German mathematician G.F.B. Riemann (1826 - 1866) observed that the frequency of prime numbers is very closely related to the behavior of an elaborate function

$$\varsigma(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots\dots\dots$$

called the *Riemann Zeta function*. The Riemann hypothesis asserts that all *interesting* solutions of the equation

$$\zeta(s) = 0$$

lie on a certain vertical straight line. This has been checked for the first 1,500,000,000 solutions. A proof that it is true for every interesting solution would shed light on many of the mysteries surrounding the distribution of prime numbers.

In his 1859 paper on the Number of Primes Less than a Given Magnitude, Bernhard Riemann (1826-1866) examined the properties of the function

$$\varsigma(s) = \sum_{n=1}^{\infty} n^{-s}$$

for s a complex number. This function is analytic for real part of s greater than 1 and is related to the prime numbers by the Euler Product Formula

$$\varsigma(s) = \prod_{p\, prime} \left(1 - p^{-s}\right)^{-1}$$

## Overview

The Riemann hypothesis implies results about the distribution of prime numbers that are in some ways as good as possible. Along with suitable generalizations, it is considered by some mathematicians to be the most important unresolved problem in pure mathematics .The Riemann hypothesis

* Dept. of Mathematics, Bhanu M.H.S.S., Biratnagar.

was computationally tested and found to be true for the first 200000001 zeros by Brent et al. (1982), covering zeros $\sigma+it$ in the region $0<t<81702130.19$. S. Wedeniwski used ZetaGrid to prove that the first trillion (1012) nontrivial zeros lie on the critical line. Gourdon (2004) then used a faster method by Odlyzko and Schönhage to verify that the first ten trillion (1013) nontrivial zeros of the $\varsigma(s)$ function lie on the critical line. This computation verifies that the Riemann hypothesis is true at least for all t less than 2.4 trillion. These results are summarized in the following table, where $g_n$ indicates a gram point.

| n | $g_n$ | source |
|---|---|---|
| $2\times10^8$ | $8.2\times10^7$ | Brent et al. (1982) |
| $10^{12}$ | $2.7\times10^{11}$ | Wedeniwski/ZetaGrid |
| $10^{13}$ | $2.4\times10^{12}$ | Gourdon (2004) |

## Extention Of Eular's Zeta Function

When studying the distribution of prime numbers Riemann extended Euler's zeta function (defined just for s with real part greater than one)

$$\varsigma(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\dots\dots\dots=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

to the entire complex plane. Riemann noted that his zeta function had trivial zeros at -2, -4, -6, ... and that all nontrivial zeros he could calculate were symmetric about the line Re(s) = 1/2. The Riemann hypothesis is that all nontrivial zeros are on this line.

## References


1. Balazard, M. and Saias, E. "The Nyman-Beurling Equivalent Form for the Riemann Hypothesis." Expos. Math. 18, 131-138, 2000.

2. Ball, W. W. R. and Coxeter, H. S. M. Mathematical Recreations and Essays, 13th ed. New York: Dover, p. 75, 1987.

3. Bombieri, E. "Problems of the Millennium: The Riemann Hypothesis." Borwein, J. and Bailey, D. Mathematics by Experiment: Plausible Reasoning in the 21st Century.

4. Brent, R. P. "On the Zeros of the Riemann Zeta Function in the Critical Strip." Math.