Derivative use of matrix

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Introduction

In Europe, the theory of matrices and determinants were first developed by "Leibnitz", while solving the solution of linear simultaneous equations.

In 1850 "Sylvester" used the word "Matrix" for array of numbers in rows and columns from which determinants is now regarded as a branch of matrix theory.

At the beginning matrix was used in solving the linear simultaneous equations, but now-a-days matrix is also used in higher mathematics while solving the differential equations. Specially, the solution of linear differential equation of second order can be obtained from matrix wit the help of differentiation.

Sylvester used the calculus on matrices to get the value of any positive integral power of a matrix.

Differentiation of matrix

If the element of matrix A are function of scalar variable t, the matrix is called a **matrix function** of t.

$$A = A(t) = \left[a_{ij}(t)\right]$$

The differential coefficient of A w.r.t. t is defined as

$$\frac{d}{dt}(A) = \left[\frac{d}{dt}(a_{ij})\right]$$

Hence the elements of differentiated matrix $\frac{dA}{dt}$ are the derivatives of the corresponding element of A.

Example : Use of matrix to solve the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0, \qquad y(0) = 1, y^1(0) = 2$$

Solution: Let

$$y = y_1, \frac{dy_1}{dx} = y_2 \qquad \text{-------(1)}$$

Or, $$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Or, $$\frac{d}{dx}\left(\frac{dy_1}{dx}\right) - 5\frac{dy_1}{dx} + 6y_1 = 0$$

Or,

$$\frac{dy_2}{dx} - 5y_2 + 6y_1 = 0 \qquad \text{-----(2)}$$

Differential equations (1) and (2) are written in matrix form

$$\frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

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The characteristic equation of

$$\begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \text{is} \begin{vmatrix} 0-\lambda & 1 \\ -6 & 5-\lambda \end{vmatrix} = 0$$

Or, $-\lambda(5-\lambda)+6=0$

Or, $\lambda^2-5\lambda+6=0 \Rightarrow \lambda=3,\ 2$

Eigen Vector for

$\lambda=3$ is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for $\lambda=2$

Let $P=\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}, P^{-1}=-\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$

$$P.e^{\lambda x}.P^{-1}=-\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} e^{3x} & 0 \\ 0 & e^{2x} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$$

$$=-\begin{bmatrix} e^{3x} & e^{2x} \\ 3e^{3x} & 2e^{2x} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$$

$$=-\begin{bmatrix} 2e^{3x}-3e^{2x} & -e^{3x}+e^{2x} \\ 6e^{3x}-6e^{2x} & -3e^{2x}+2e^{2x} \end{bmatrix}$$

$$=\begin{bmatrix} 3e^{2x}-2e^{3x} & e^{3x}-e^{2x} \\ 6e^{2x}-6e^{3x} & 3e^{3x}-2e^{2x} \end{bmatrix}$$

On applying the initial conditions

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} 3e^{2x}-2e^{3x} & e^{3x}-e^{2x} \\ 6e^{2x}-6e^{3x} & 3e^{3x}-2e^{2x} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$=\begin{bmatrix} 3e^{2x}-2e^{3x}+2e^{3x}-2e^{2x} \\ 6e^{2x}-6e^{3x}+6e^{3x}-4e^{2x} \end{bmatrix}$$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} e^{2x} \\ 2e^{2x} \end{bmatrix}$$

Sylvester's Theorem

Let

$P(A)=C_0A^n+C_1A^{n-1}+C_2A^{n-2}+\dots\dots\dots+C_{n-1}A+C_nI$ and $|\lambda I-A|=f(\lambda)$ and adjoint matrix of $[\lambda I-A]=[f(\lambda)]$

$$Z(\lambda)=\frac{[f(\lambda)]}{f^1(\lambda)}=\frac{\text{Adjoint matrix }[\lambda I-A]}{f^1(A)}$$

Then according to Sylvester's theorem

$$P(A)=P(\lambda_1).Z(\lambda_1)+P(\lambda_2).Z(\lambda_2)+P(\lambda_3).Z(\lambda_3)+\dots\dots\dots$$

$$=\sum_{r=1}^{n}P(\lambda_r).Z(\lambda_r)$$

Example : If $A=\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, find A^{50}

Solution :

$$f(\lambda)=[\lambda I-A]=\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}-\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}=\begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-3 \end{bmatrix}=0$$

Or,

$f(\lambda)=(\lambda-1)(\lambda-3)=0$ or, $\lambda_1=1,\ \lambda_2=3$

$f(\lambda)=\lambda^2-4\lambda+3,\ f^1(\lambda)=2\lambda-4$

$f^1(1)=2-4=-2,\ f^1(3)=6-4=2$

$$Z(\lambda_1)=Z(1)=\frac{[f(1)]}{f^1(1)}=-\frac{1}{2}\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$Z(\lambda_2)=Z(3)=\frac{[f(3)]}{f^1(3)}=\frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

By using Sylvester's theorem

$$P(A)=P(\lambda_1).Z(\lambda_1)+P(\lambda_2).Z(\lambda_2)$$

$$A^{50}=P(\lambda_1).Z(\lambda_1)+P(\lambda_2).Z(\lambda_2)$$

$$=\lambda_1^{50}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}+\lambda_2^{50}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$=1^{50}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}+3^{50}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\0 & 3^{50}\end{bmatrix}$$

$$\therefore A^{50}=\begin{bmatrix}1 & 0\\0 & 3^{50}\end{bmatrix}$$

Conclusion

It is important to note that Matrix itself is an important branch of mathematics and its use in different field of mathematics gives miracle results. Sometimes for the solution of linear differential equation of second order and the value of positive integral power of a matrix have no alternatives than the use of matrix with the help of differentiation.

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