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On zero-free regions for the derivative of a polynomial

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Abstract

Let P_n denote the set of all polynomials of the form $p(z) = z\prod_{j=1}^{n-1}(z - z_j)$ with $|z_j| \geq 1, \quad 1 \leq j \leq n-1$. In this paper we shall obtain some zero-free regions for the derivative of a polynomial.

Keywords: Zero-free regions; Critical points; Sendov's Conjecture.

1. Introduction

Let us suppose that p(z) is an n^{th} degree polynomial which has all its zeros in the unit disk $|z| \leq 1$, then all the critical points of p(z) also lie in the same disk $|z| \leq 1$.This is in fact the well-known Theorem which was implied in a note of Gauss dated 1836 and proved explicitly by Lucas dated 1874 (see also Marden [1].

Now instead of considering the relative position of all the zeros and critical points of p(z), let us choose any one zero z_o of p(z) and ask: At most how far from z_o does the nearest critical point lie ? A possible answer to this question is given by the following :**Conjecture**: " If p(z) is an n^{th} degree polynomial having all its zeros in the unit disk $|z| \leq 1$ and if z_o is any one such zero, then at least one critical point of p(z) lie in the disk $|z - z_0| \leq 1$." This conjecture was included in the collection of Research Problems in Function Theory published in 1967 by professor Hayman [2], (see also [3]). Since it had been brought to Hayman's attention by professor Ilyeff. It became known as "Ilyeff's Conjecture". Actually conjecture was due to a Bulgarian mathematician B. Sendov. In connection with this conjecture Brown [4] posed the following problem.

Let Q_n denote the set of all complex polynomials of the form $p(z)=z\prod_{j=1}^{n-1}\left(z-z_j\right)$ with $\left|z_j\right|\geq 1,\quad 1\leq j\leq n-1.$ Find the best constant C_n such that $p'(z)$ does not vanish in $\left|z\right|\leq C_n$ for all $p\in Q_n$.

Brown observed that if $p(z)=z(z-1)^{n-1}$ then $p'\left(\frac{1}{n}\right)=0$ and conjectured that $C_n=\frac{1}{n}$.

Recently Aziz and Zargar [5] settled this conjecture.

Theorem1.1. Let $p(z)=z\prod_{k=1}^{n-1}\left(z-z_k\right)$ be a polynomial of degree n with $\left|z_k\right|\geq 1,\quad 1\leq k\leq n-1,$ then $p'(z)$ does not vanish in the disk $\left|z\right|<\frac{1}{n}$.

The result is best possible for the polynomial $p(z)=z(z-e^{i\theta})^{n-1}, 0\leq\theta<2\pi$.

First we shall prove the following interesting result which provides the zero free regions for the second derivative of polynomial

$$p(z)=z^m\prod_{k=1}^{n-m}\left(z-z_j\right)$$

Theorem 1.2. If $p(z)=z^m\prod_{j=1}^{n-m}\left(z-z_j\right)$ where $\left|z_j\right|\geq 1,\ j=1,2,.........n-m,$ then the polynomial $p''(z)$ does not vanish in

$$0<\left|z\right|<\frac{m(m-1)}{n(n-1)}.$$

Taking m = 2 we get

Corollary 1. If $p(z)=z^2\prod_{j=1}^{n-2}\left(z-z_j\right)$ where $\left|z_j\right|\geq 1,\ j=1,2,.........n-2,$ then the polynomial $p''(z)$ does not vanish in

$$0<\left|z\right|<\frac{2}{n(n-1)}.$$

It is clearly of interest to known that a zero free region for the polynomial $p^m(z)$ where

$$p(z)=z^m\prod_{j=1}^{n-m}\left(z-z_j\right)$$

In this direction, we prove the following interesting results:

Theorem1.3. Let

$$p(z)=z^m\prod_{j=1}^{n-m}\left(z-z_j\right)$$

be a polynomial of degree n, with $\left|z_j\right|\geq 1,\ j=1,2,.........n-m,$ then the polynomial $p^m(z)$ does not vanish in the disk

$$|z|<\frac{m!}{n(n-1)....(n-m+1)}.$$

Remark 1. If m=1,then we get Theorem 1.1.

For the proofs of these theorems we need the following result which is due to Aziz and Zagar [5].

Lemma: Let $p(z)=z^m\prod_{j=1}^{n-m}\left(z-z_j\right)$ where $\left|z_j\right|\geq 1,\quad 1\leq j\leq n-m,$ then $p'(z)$ does not vanish in $0<|z|<\frac{m}{n}.$

2. Proofs of Theorems

Proof of Theorem 1.2. We write,

$$p(z)=z^mQ(z)$$

where

$$Q(z)=\prod_{j=1}^{n-m}\left(z-z_j\right),\left|z_j\right|\geq 1,\ j=1,2,......n-m.$$

By above lemma, the polynomial

$p'(z)=z^mQ'(z)+mz^{m-1}Q(z)$

$=z^{m-1}R(z)$,

where

$$R(z)=zQ'(z)+mQ(z),$$

does not vanish in $0<|z|<\frac{m}{n}.$

Replacing zby $\frac{m}{n}z,$ it follows that the polynomial,

$$S(z)=p'\left(\frac{m}{n}z\right)$$

$$=\left(\frac{m}{n}\right)^{m-1}z^{m-1}R\left(\frac{m}{n}z\right)$$

does not vanish in $0<|z|<1,$so that all the zeros of $R\left(\frac{m}{n}z\right)$ lie in $|z|\geq 1.$

Using the above lemma again and noting that S(z) is a polynomial of degree n-1, it follows that $S'(z)$ does not vanish in

$$0 < |z| < \frac{m-1}{n-1}.$$

Or equivalently,

$$p''\left(\frac{m}{n}z\right)$$

does not vanish in

$$0 < |z| < \frac{m-1}{n-1}$$

Replacing z by $\frac{m}{n}z$, it follows that

$$\begin{aligned} p''(z) &= z^{m-1}R'(z) + (m-1)z^{m-2}R(z) \\ &= z^{m-2}\left(zR'(z) + (m-1)R(z)\right) \\ &= z^{m-2}T(z) \end{aligned}$$

where

$$T(z) = zR'(z) + (m-1)zR(z)$$

does not vanish in

$$0 < |z| < \frac{m-1}{n-1}.$$

This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. By hypothesis,

$$p(z) = z^m Q(z)$$

where

$$Q(z) = \prod_{j=1}^{n-m}\left(z - z_j\right), |z_j| \geq 1,\ j = 1,2,......n-m\ .$$

By the above lemma, the polynomial $p'(z)$ does not vanish in

$$0 < |z| < \frac{m}{n}.$$

Therefore Theorem 1.2 yeilds that

$$p''(z) = z^{n-2}T(z)\ ,$$

where

$$T(z) = zR'(z) + (m-1)zR(z)$$

does not vanish in

$$0 < |z| < \frac{m(m-1)}{n(n-1)}.$$

Replacing z by $\frac{m(m-1)}{n(n-1)}z$, it follows that

$$U(z) = p''\left(\frac{m(m-1)}{n(n-1)}z\right)$$
$$= \left(\frac{m(m-1)}{n(n-1)}z\right)^{n-2} T\left(\frac{m(m-1)}{n(n-1)}z\right)$$

does not vanish in $0 < |z| < 1$, so that all the zeros of $T\left(\frac{m(m-1)}{n(n-1)}z\right)$ lie in $|z| \geq 1$.Applying the above lemma again and noting that U(z) is a polynomial of degree n-2, thus it implies that $U'(z)$ does not vanish in $0 < |z| < \frac{m-2}{n-2}$.

Or equivalently,

$$p'''\left(\frac{m(m-1)}{n(n-1)}z\right)$$

does not vanish in

$$0 < |z| < \frac{m-2}{n-2}.$$

Replacing z by $\frac{n(n-1)}{m(m-1)}z$ it follows that,

$$p'''(z) = z^{n-2}T'(z) + (n-2)z^{n-3}T(z)$$
$$= z^{n-3}\left(zT'(z) + (n-2)T(z)\right)$$
$$= z^{n-3}V(z),$$

where

$$V(z) = zT'(z) + (n-2)T(z)$$

does not vanish in $0 < |z| < \frac{m(m-1)(m-2)}{n(n-1)(n-2)}$.

In a similar way we see that the polynomial

$$p^{iv}(z) = z^{n-3}V'(z) + (n-3) + (n-3)z^{n-4}V(z)$$
$$= z^{n-4}W(z)$$

where,

$$W(z) = zV'(z) + (n-3)V(z)$$

does not vanish in

$$0 < |z| < \frac{m(m-1)(m-2)(m-3)}{n(n-1)(n-2)(n-3)}.$$

Proceeding in this way and noting that m and n are positive integers it follows that the polynomial does not vanish in

$$|z| < \frac{m(m-1).......2.1}{n(n-1)....(n-m+1)} = \frac{m!}{n(n-1).....(n-m+1)}$$

Which proves Theorem 1.3.

References

[1] M. Marden, Geometry of polynomials, Math. Survey's No.3, American Math.Soc. (1966).

[2] W. K. Hayman, Research Problems in Function Theory, London (1967) pp. 1-25.

[3] Q. I. Rahman and G. Schmesser's Analytic Theory of polynomials, Oxford University Press,New York (2002).

[4] J. E. Brown, On theIlief-Sendov Conjecture, Pacific J.Math.135 (1988) 223-232. http://dx.doi.org/10.2140/pjm.1988.135.223

[5] A. Aziz and B.A. Zargar, On the critical points of a polynomial, Bull. Austral. Math. Soc.57 (1998) 173-174. http://dx.doi.org/10.1017/S000497270003152X