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Generalizations of some Enestrom-Kakeya type results

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Abstract

In this paper we give interesting generalizations of some well-known Enestrom-Kakeya type results on the location of zeros of a complex polynomial under less restrictive conditions on the coefficients of the polynomial.

Keywords: Bound; Coefficient; Polynomial; Zeros.

1. Introduction

Regarding the zeros of a polynomial with real and positive coefficients, we have the following result known as the Enestrom-Kakeya Theorem [1, 2, 3].

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 > 0 .$$

Then all the zeros of $P(z)$ lie in $|z| \leq 1$.

Various extensions and generalizations of this result are available in the literature. Joyal et al [4] proved the following generalization of Theorem A:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 .$$

Then $P(z)$ has all its zeros in the disk

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|} .$$

As a generalization of Theorems A and B, Aziz and Zargar [5] proved the following:

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \geq 1$

$$ka_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 .$$

Then $P(z)$ has all its zeros in the disk

$$|z+k-1| \le \frac{ka_n - a_0 + |a_0|}{|a_n|} .$$

Shah et al [6] extended Theorem B to polynomials with complex coefficients and proved the following result:

Theorem D: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a complex polynomial of degree n with $\text{Re}(a_j)=\alpha_j$ and $\text{Im}(a_j)=\beta_j$ for $j=0,1,2,......,n$, such that for some $k \ge 1$

$$k\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \alpha_0$$
$$\beta_n \ge \beta_{n-1} \ge ...... \ge \beta_1 \ge \beta_0 > 0.$$

Then $P(z)$ has all its zeros in the disk

$$\left|z+\frac{\alpha_n}{a_n}(k-1)\right| \le \frac{k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n}{|a_n|} .$$

Liman et al [7] proved the following generalization of Theorem D:

Theorem E: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a complex polynomial of degree n with $\text{Re}(a_j)=\alpha_j$ and $\text{Im}(a_j)=\beta_j$ for $j=0,1,2,......,n, a_n \ne 0$. If for some positive integer $\lambda \le n$ and $k \ge 1$

$$k^{n-\lambda+1}\alpha_n \ge k^{n-\lambda}\alpha_{n-1} \ge k^{n-\lambda-1}\alpha_{n-2} \ge ...... \ge k^2\alpha_{\lambda+1} \ge k\alpha_\lambda \ge \alpha_{\lambda-1} \ge ...... \ge \alpha_1 \ge \alpha_0 ,$$
$$\beta_n \ge \beta_{n-1} \ge .... \ge \beta_1 \ge \beta_0 > 0 ,$$

then all the zeros of $P(z)$ lie in

$$\left|z+\frac{\alpha_n}{a_n}(k-1)\right| \le \frac{\alpha_n - \alpha_0 + |\alpha_0| + (k-1)\left(\sum_{j=\lambda}^{n}(\alpha_j + |\alpha_j|) - |\alpha_n|\right) + \beta_n}{|a_n|} .$$

In the same paper they proved the following result also.

Theorem F: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a complex polynomial of degree n with $\text{Re}(a_j)=\alpha_j$ and $\text{Im}(a_j)=\beta_j$ for $j=0,1,2,......,n, a_n \ne 0$. If for some positive integer $\lambda \le n$, $k \ge 1$ and $\rho \ge 1$,

$$k^{n-\lambda+1}\alpha_n \ge k^{n-\lambda}\alpha_{n-1} \ge k^{n-\lambda-1}\alpha_{n-2} \ge ...... \ge k^2\alpha_{\lambda+1} \ge k\alpha_\lambda \ge \alpha_{\lambda-1} \ge ...... \ge \alpha_1 \ge \alpha_0 ,$$
$$\rho\beta_n \ge \beta_{n-1} \ge .... \ge \beta_1 \ge \beta_0 > 0 ,$$

then all the zeros of $P(z)$ lie in

$$\left|z+\frac{k\alpha_n + i\rho\beta_n}{a_n} - 1\right| \le \frac{\alpha_n - \alpha_0 + |\alpha_0| + (k-1)\left(\sum_{j=\lambda}^{n}(\alpha_j + |\alpha_j|) - |\alpha_n|\right) + \rho\beta_n}{|a_n|} .$$

Recently Gulshan Singh [8] proved the following generalization of Theorems E and F:

Theorem G: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a complex polynomial of degree n with $a_j=\alpha_j+i\beta_j,\ j=0,1,2,......,n$, where α_j and β_j are real numbers. If for some positive integers $\lambda,\mu\le n$ and for some real numbers $0<\tau_1\le 1, 0<\tau_2\le 1, k\ge 1$,

$$k^{n-\lambda+1}\alpha_n\ge k^{n-\lambda}\alpha_{n-1}\ge k^{n-\lambda-1}\alpha_{n-2}\ge......\ge k^2\alpha_{\lambda+1}\ge k\alpha_\lambda\ge\alpha_{\lambda-1}\ge......\ge\alpha_1\ge\tau_1\alpha_0,$$

$$k^{n-\mu+1}\beta_n\ge k^{n-\mu}\beta_{n-1}\ge k^{n-\mu-1}\beta_{n-2}\ge......\ge k^2\beta_{\mu+1}\ge k\beta_\mu\ge\beta_{\mu-1}\ge......\ge\beta_1\ge\tau_2\beta_0,$$

then all the zeros of $P(z)$ lie in

$$|z+k-1|\le\frac{1}{|a_n|}\Big[(\alpha_n+\beta_n)-(\tau_1\alpha_0+\tau_2\beta_0)+(1-\tau_1)|\alpha_0|+(1-\tau_2)|\alpha_0|+(|\alpha_0|+|\beta_0|)$$
$$+(k-1)\sum_{j=\lambda}^{n}(\alpha_j+|\alpha_j|)+\sum_{j=\mu}^{n}(\beta_j+|\beta_j|)-|\alpha_n|-|\beta_n|\Big]^n.$$

In this paper we prove the following result which not only generalizes the above results but also gives many other results for different values of the parameters.

2. Theorems and Proofs

Theorem 1. Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a complex polynomial of degree n with $a_j=\alpha_j+i\beta_j,\ j=0,1,2,......,n$, where α_j and β_j are real numbers. If for some positive integers $\lambda,\mu\le n$ and for some real numbers $0<\tau_1\le 1, 0<\tau_2\le 1, k_1\ge 1, k_2\ge 1$,

$$k_1^{n-\lambda+1}\alpha_n\ge k_1^{n-\lambda}\alpha_{n-1}\ge k_1^{n-\lambda-1}\alpha_{n-2}\ge......\ge k_1^2\alpha_{\lambda+1}\ge k_1\alpha_\lambda\ge\alpha_{\lambda-1}\ge......\ge\alpha_1\ge\tau_1\alpha_0,$$

$$k_2^{n-\mu+1}\beta_n\ge k_2^{n-\mu}b_{n-1}\ge k_2^{n-\mu-1}\beta_{n-2}\ge......\ge k_2^2\beta_{\mu+1}\ge k_2\beta_\mu\ge\beta_{\mu-1}\ge......\ge\beta_1\ge\tau_2\beta_0,$$

then all the zeros of $P(z)$ lie in

$$\left|z+\frac{(k_1-1)\alpha_n+i(k_2-1)\beta_n}{a_n}\right|\le\frac{1}{|a_n|}\Big[(\alpha_n+\beta_n)-(\tau_1\alpha_0+\tau_2\beta_0)+(1-\tau_1)|\alpha_0|$$
$$+(1-\tau_2)|\beta_0|+(|\alpha_0|+|\beta_0|)$$
$$+(k-1)\left(\sum_{j=\lambda}^{n}(\alpha_j+|\alpha_j|)\right)-(k_1-1)|\alpha_n|$$
$$+(k_2-1)\left(\sum_{j=\mu}^{n}(\beta_j+|\beta_j|)\right)-(k_2-1)|\beta_n|\ \Big].$$

Taking $k_1=k_2=k,\ \tau_1=\tau_2=\tau,$ in Theorem 1 we get Theorem G. Taking $k_1=k, k_2=1,\ \tau_1=\tau_2=1,\ and\ \beta_0\ge 0$ in Theorem 1, we get Theorem E. Taking $k_1=k,\ k_2=\mu,\mu=n,\ \tau_1=\tau_2=1\ and\ \beta_0\ge 0$ in Theorem 1 we get Theorem F of Liman et al.Taking $\lambda=n=\mu$ in Theorem 1,we get a Theorem of Gulzar [9,Theorem 1].

Taking $k_1=k_2=1=\tau_1=\tau_2$ in Theorem 1 we get the following interesting result:

Corollary 1. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a complex polynomial of degree n with $\mathrm{Re}(a_j)=\sim_j . \mathrm{Im}(a_j)= \ _j \ for \ j=0,1,2,......,n,$ such that

$$\sim_n \geq \sim_{n-1} \geq ...... \geq \sim_1 \geq \sim_0 ,$$
$$\ _n \geq \ _{n-1} \geq ...... \geq \ _1 \geq \ _0 .$$

Then $P(z)$ has all its zeros in the disk

$$|z| \leq \frac{1}{|a_n|}(\sim_n + \ _n)$$

Taking a_j real, that is $\ _j=0$ for j=0,1,2,……..,n and $\sim_0 > 0$,Corollary 1 reduces to Enestrom-Kakeya Theorem.

Taking ª = « in Theorem 1, we get the following result:.

Corollary 2. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a complex polynomial of degree n with $\mathrm{Re}(a_j)=\sim_j . \mathrm{Im}(a_j)= \ _j \ for \ j=0,1,2,......,n, \ a_n \neq 0$. If for some positive integer

$$ª \leq n, 0 < °_1 \leq 1, \ 0 < °_2 \leq 1, \ k_1 \geq 1, \ k_2 \geq 1,$$
$$k_1^{n-ª+1}\sim_n \geq k_1^{n-ª}\sim_{n-1} \geq k_1^{n-ª-1}\sim_{n-2} \geq ...... \geq k_1^{2}\sim_{ª+1} \geq k_1\sim_ª \geq \sim_{ª-1} \geq ...... \geq \sim_1 \geq °_1\sim_0 ,$$
$$k_2^{n-ª+1} \ _n \geq k_2^{n-ª} \ _{n-1} \geq k_2^{n-ª-1} \ _{n-2} \geq ...... \geq k_2^{2} \ _{ª+1} \geq k_2 \ _ª \geq \ _{ª-1} \geq ...... \geq \ _1 \geq °_2 \ _0 ,$$

then all the zeros of $P(z)$ lie in

$$\left| z + \frac{(k_1-1)\sim_n + i(k_2-1) \ _n}{a_n} \right| \leq \frac{1}{|a_n|}\Big[(\sim_n + \ _n) - (°_1\sim_0 + °_2 \ _0) + (1-°_1)|\sim_0|$$
$$+(1-°_2)| \ _0| + (|\sim_0| + | \ _0|)$$
$$+(k-1)\left(\sum_{j=ª}^{n}(\sim_j + |\sim_j|)\right) - (k_1-1)|\sim_n|$$
$$+(k_2-1)\left(\sum_{j=ª}^{n}(\ _j + | \ _j|)\right) - (k_2-1)| \ _n| \ \Big].$$

Many other results may be deduced from Theorem 1 for different values of parameters.

Proof of Theorem 1: Consider the polynomial

$$
\begin{aligned}
F(z) &= (1-z)p(z) \\
&= (1-z)(a_n z^n + a_{n-1}z^{n-1} + \ldots\ldots\ldots\ldots + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots\ldots\ldots + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} + (\sim_n - \sim_{n-1})z^n + \ldots\ldots\ldots + (\sim_1 - \sim_0)z + \sim_0 \\
&\quad + i\{(\ \ _n - \ \ _{n-1})z^n + \ldots\ldots\ldots + (\ \ _1 - \ \ _0)z + \ \ _0\}. \\
&= -a_n z^{n+1} - (k_1 - 1)\sim_n z^n + (k_1\sim_n - \sim_{n-1})z^n + (k_1\sim_{n-1} - \sim_{n-2})z^{n-1} + \ldots\ldots \\
&\quad + (k_1\sim_{\text{ª}+1} - \sim_{\text{ª}})z^{\text{ª}+1} + (k_1\sim_{\text{ª}} - \sim_{\text{ª}-1})z^{\text{ª}} + (\sim_{\text{ª}-1} - \sim_{\text{ª}-2})z^{\text{ª}-1} + (\sim_{\text{ª}-2} - \sim_{\text{ª}-3})z^{\text{ª}-2} + \ldots\ldots \\
&\quad + (\sim_2 - \sim_1)z^2 + (\sim_1 - {}^{\circ}{}_1\sim_0)z + ({}^{\circ}{}_1 - 1)\sim_0 z + \sim_0 - (k_1 - 1)\left(\sim_{n-1}z^{n-1} + \ldots\ldots + \sim_{\text{ª}+1}z^{\text{ª}+1} + \sim_{\text{ª}}z^{\text{ª}}\right) \\
&\quad + i\{-(k_2 - 1)\ \ _n z^n + (k_2\ \ _n - \ \ _{n-1})z^n + (k_2\ \ _{n-1} - \ \ _{n-2})z^{n-1} + \ldots\ldots \\
&\quad + (k_2\ \ _{\text{«}+1} - \ \ _{\text{«}})z^{\text{«}+1} + (k_2\ \ _{\text{«}} - \ \ _{\text{«}-1})z^{\text{«}} + (\ \ _{\text{«}-1} - \ \ _{\text{«}-2})z^{\text{«}-1} \\
&\quad + (\ \ _{\text{«}-2} - \ \ _{\text{«}-3})z^{\text{«}-2} + \ldots\ldots + (\ \ _2 - \ \ _1)z^2 + (\ \ _1 - {}^{\circ}{}_2\ \ _0)z + ({}^{\circ}{}_2 - 1)\ \ _0 z + \ \ _0 \\
&\quad - (k_2 - 1)\left(\ \ _{n-1}z^{n-1} + \ldots\ldots + \ \ _{\text{«}+1}z^{\text{«}+1} + \ \ _{\text{«}}z^{\text{«}}\right)\}]
\end{aligned}
$$

$$
\begin{aligned}
&= z^n[\ -a_n z - (k_1 - 1)\sim_n - i(k_2 - 1)\ \ _n + (k_1\sim_n - \sim_{n-1}) + (k_1\sim_{n-1} - \sim_{n-2})\frac{1}{z} + \ldots\ldots\ldots \\
&\quad + (k_1\sim_{\text{ª}+1} - \sim_{\text{ª}})\frac{1}{z^{n-\text{ª}-1}} + (k_1\sim_{\text{ª}} - \sim_{\text{ª}-1})\frac{1}{z^{n-\text{ª}}} + (\sim_{\text{ª}-1} - \sim_{\text{ª}-2})\frac{1}{z^{n-\text{ª}+1}} \\
&\quad + (\sim_{\text{ª}-2} - \sim_{\text{ª}-3})\frac{1}{z^{n-\text{ª}+2}} + \ldots\ldots + (\sim_2 - \sim_1)\frac{1}{z^{n-2}} + (\sim_1 - {}^{\circ}{}_1\sim_0)\frac{1}{z^{n-1}} \\
&\quad + ({}^{\circ}{}_1 - 1)\sim_0\frac{1}{z^{n-1}} + \sim_0\frac{1}{z^n} - (k_1 - 1)\left(\sim_{n-1}\frac{1}{z} + \ldots\ldots + \sim_{\text{ª}}\frac{1}{z^{n-\text{ª}}}\right) \\
&\quad + i\{(k_2\ \ _n - \ \ _{n-1}) + (k_2\ \ _{n-1} - \ \ _{n-2})\frac{1}{z} + \ldots\ldots \\
&\quad + (k_2\ \ _{\text{«}+1} - \ \ _{\text{«}})\frac{1}{z^{n-\text{«}-1}} + (k_2\ \ _{\text{«}} - \ \ _{\text{«}-1})\frac{1}{z^{n-\text{«}}} + (\ \ _{\text{«}-1} - \ \ _{\text{«}-2})\frac{1}{z^{n-\text{«}+1}} \\
&\quad + (\ \ _{\text{«}-2} - \ \ _{\text{«}-3})\frac{1}{z^{n-\text{«}+2}} + \ldots\ldots + (\ \ _2 - \ \ _1)\frac{1}{z^{n-2}} + (\ \ _1 - {}^{\circ}{}_2\ \ _0)\frac{1}{z^{n-1}} \\
&\quad + ({}^{\circ}{}_2 - 1)\ \ _0\frac{1}{z^{n-1}} + \frac{\ \ _0}{z^n} - (k_2 - 1)\left(\ \ _{n-1}\frac{1}{z} + \ldots\ldots + \ \ _{\text{«}}\frac{1}{z^{n-\text{«}}}\right)\}]
\end{aligned}
$$

For |z|>1 so that $\frac{1}{|z|^{n-j}} < 1,\ j = 0,1,2,\ldots\ldots\ldots, n$ .We have by using the hypothesis,

$$
\begin{aligned}
|F(z)| \geq |z|^n \Big[& |a_n z + (k_1-1)\sim_n + i(k_2-1)\ \ _n| - \Big\{ |k_1\sim_n - \sim_{n-1}| + |k_1\sim_{n-1} - \sim_{n-2}| \frac{1}{|z|} \\
& + |k_1 \sim_{\text{ª}} - \sim_{\text{ª}-1}| \frac{1}{|z|^{n-\text{ª}}} + |\sim_{\text{ª}-1} - \sim_{\text{ª}-2}| \frac{1}{|z|^{n-\text{ª}+1}} + |\sim_{\text{ª}-2} - \sim_{\text{ª}-3}| \frac{1}{|z|^{n-\text{ª}+2}} + \ldots\ldots. \\
& + |\sim_2 - \sim_1| \frac{1}{|z|^{n-2}} + |\sim_1 - {}^\circ{}_1 \sim_0| \frac{1}{|z|^{n-1}} + |{}^\circ{}_1 - 1| |\sim_0| \frac{1}{|z|^{n-1}} + |\sim_0| \frac{1}{|z|^{n-1}} \\
& - (k_1-1)\left(|\sim_{n-1}| \frac{1}{|z|} + \ldots\ldots.. + |\sim_{\text{ª}}| \frac{1}{|z|^{n-\text{ª}}} \right) + |k_2 \ \ _n - \ \ _{n-1}| + |k_2 \ \ _{n-1} - \ \ _{n-2}| \frac{1}{|z|} + \ldots\ldots. \\
& + |k_2 \ \ _{\text{«}+1} - \ \ _{\text{«}}| \frac{1}{|z|^{n-\text{«}-1}} + |k_2 \ \ _{\text{«}} - \ \ _{\text{«}-1}| \frac{1}{|z|^{n-\text{«}}} + |\ \ _{\text{«}-1} - \ \ _{\text{«}-2}| \frac{1}{|z|^{n-\text{«}+1}} \\
& + |\ \ _{\text{«}-2} - \ \ _{\text{«}-3}| \frac{1}{|z|^{n-\text{«}+2}} + \ldots\ldots.. + |\ \ _2 - \ \ _1| \frac{1}{|z|^{n-2}} + |\ \ _1 - {}^\circ{}_2 \ \ _0| \frac{1}{|z|^{n-1}} \\
& + |{}^\circ{}_2 - 1| |\ \ _0| \frac{1}{|z|^{n-1}} + \frac{|\ \ _0|}{|z|^n} - (k_2-1)\left(|\ \ _{n-1}| \frac{1}{|z|} + \ldots\ldots.. + |\ \ _{\text{«}}| \frac{1}{|z|^{n-\text{«}}} \right) \Big\} \Big] \\
\geq |z|^n \Big[& |a_n z + (k_1-1)\sim_n + i(k_2-1)\ \ _n| - \{ k_1\sim_n - \sim_{n-1} + k_1 \sim_{n-1} - \sim_{n-2} \\
& + k_1\sim_{\text{ª}} - \sim_{\text{ª}-1} + \sim_{\text{ª}-1} - \sim_{\text{ª}-2} + \sim_{\text{ª}-2} - \sim_{\text{ª}-3} + \ldots\ldots. \\
& + \sim_1 - {}^\circ{}_1 \sim_0 + (1 - {}^\circ{}_1)|\sim_0| + |\sim_0| \\
& + (k_1-1)(|\sim_{n-1}| + \ldots\ldots.. + |\sim_{\text{ª}}|) + k_2 \ \ _n - \ \ _{n-1} + k_2 \ \ _{n-1} - \ \ _{n-2} + \ldots\ldots. \\
& + k_2 \ \ _{\text{«}+1} - \ \ _{\text{«}} + k_2 \ \ _{\text{«}} - \ \ _{\text{«}-1} + \ \ _{\text{«}-1} - \ \ _{\text{«}-2} \\
& + \ \ _{\text{«}-2} - \ \ _{\text{«}-3} + \ldots\ldots.. + \ \ _2 - \ \ _1 + \ \ _1 - {}^\circ{}_2 \ \ _0 \\
& + (1 - {}^\circ{}_2)|\ \ _0| + |\ \ _0| - (k_2-1)(|\ \ _{n-1}| + \ldots\ldots.. + |\ \ _{\text{«}}|) \} \Big]
\end{aligned}
$$

$$
\begin{aligned}
= |z|^n \Big[& |a_n z + (k_1-1)\sim_n + i(k_2-1)\ \ _n| - \{ (\sim_n + \ \ _n) - ({}^\circ{}_1 \sim_0 + {}^\circ{}_2 \ \ _0) + (1 - {}^\circ{}_1)|\sim_0| + (1 - {}^\circ{}_2)|\ \ _0| \\
& + |\sim_0| + |\ \ _0| + (k_1-1)\left(\sum_{j=\text{ª}}^{n} (\sim_j + |\sim_j|) \right) - (k_1-1)|\sim_n| + (k_2-1)\left(\sum_{j=\text{«}}^{n} (\ \ _j + |\ \ _j|) \right) - (k_2-1)|\ \ _n| \} \Big] \\
> 0 &
\end{aligned}
$$

if

$$
\begin{aligned}
|a_n z + (k_1-1)\sim_n + i(k_2-1)\ \ _n| > & (\sim_n + \ \ _n) - ({}^\circ{}_1 \sim_0 + {}^\circ{}_2 \ \ _0) + (1 - {}^\circ{}_1)|\sim_0| + (1 - {}^\circ{}_2)|\ \ _0| \\
& + |\sim_0| + |\ \ _0| + (k_1-1)\left(\sum_{j=\text{ª}}^{n} (\sim_j + |\sim_j|) \right) - (k_1-1)|\sim_n| \\
& + (k_2-1)\left(\sum_{j=\text{«}}^{n} (\ \ _j + |\ \ _j|) \right) - (k_2-1)|\ \ _n|
\end{aligned}
$$

that is, if

$$\left|z+\frac{(k_1-1)\alpha_n+i(k_2-1)\beta_n}{a_n}\right|>\frac{1}{|a_n|}\Big[(\alpha_n+\beta_n)-(\tau_1\alpha_0+\tau_2\beta_0)+(1-\tau_1)|\alpha_0|+(1-\tau_2)|\beta_0|+(|\alpha_0|+|\beta_0|)$$
$$+(k-1)\left(\sum_{j=\lambda}^{n}(\alpha_j+|\alpha_j|)\right)-(k_1-1)|\alpha_n|$$
$$+(k_2-1)\left(\sum_{j=\lambda}^{n}(\beta_j+|\beta_j|)\right)-(k_2-1)|\beta_n|$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in

$$\left|z+\frac{(k_1-1)\alpha_n+i(k_2-1)\beta_n}{a_n}\right|\le\frac{1}{|a_n|}\Big[(\alpha_n+\beta_n)-(\tau_1\alpha_0+\tau_2\beta_0)+(1-\tau_1)|\alpha_0|+(1-\tau_2)|\beta_0|+(|\alpha_0|+|\beta_0|)$$
$$+(k-1)\left(\sum_{j=\lambda}^{n}(\alpha_j+|\alpha_j|)\right)-(k_1-1)|\alpha_n|$$
$$+(k_2-1)\left(\sum_{j=\lambda}^{n}(\beta_j+|\beta_j|)\right)-(k_2-1)|\beta_n|\Big]$$

Since those zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality, it follows that all the zeros of F(z) and hence of p(z) lie in

$$\left|z+\frac{(k_1-1)\alpha_n+i(k_2-1)\beta_n}{a_n}\right|\le\frac{1}{|a_n|}\Big[(\alpha_n+\beta_n)-(\tau_1\alpha_0+\tau_2\beta_0)+(1-\tau_1)|\alpha_0|+(1-\tau_2)|\beta_0|+(|\alpha_0|+|\beta_0|)$$
$$+(k-1)\left(\sum_{j=\lambda}^{n}(\alpha_j+|\alpha_j|)\right)-(k_1-1)|\alpha_n|$$
$$+(k_2-1)\left(\sum_{j=\lambda}^{n}(\beta_j+|\beta_j|)\right)-(k_2-1)|\beta_n|\Big]$$

This completes the proof of Theorem 1.

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