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On contact conformal curvature tensor in trans-Sasakian manifolds

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Abstract

The purpose of this paper is to study some results on contact conformal curvature tensor in trans-Sasakian manifolds. Contact conformally flat trans-Sasakian manifold, ξ-contact conformally flat trans-Sasakian manifold and curvature conditions $C_0(\xi, X).S = 0$ and $C_0(\xi, X).C_0 = 0$ are studied with some interesting results. Finally, we study an example of 3-dimensional trans-Sasakian manifold.

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1. Introduction

In 1978, Gray and Hervella [1] studied on the sixteen classes of almost Hermitian manifolds and their linear invariants. They considered unitary group $U(n)$ on a certain space W and studied that the representation of $U(n)$ on W has four irreducible components, $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$. From these four components sixteen different invariants subspaces were obtained. Among four components $W_3 \oplus W_4$ corresponds to the class of Hermitian manifolds. Oubina [2] studied a new class of almost contact metric structure, called trans-Sasakian which is an analogue of a locally conformal Kaehler structure on an almost Hermitian manifold. An almost contact metric structure (φ, ξ, η, g) (where φ is a (1, 1) tensor field, ξ is a vector field, η is a 1-form and g is a compatible Rimannian metric) on M is trans-Sasakian [2] if $(M \times \mathbb{R}, J, G)$ belongs to the class W_4, where J is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right), \tag{1.1}$$

for any vector field X on M, where G is the product metric on $M \times \mathbb{R}$. Trans-Sasakian manifold is the trans-Sasakian structure of type (α, β), where α and β are smooth functions on M. Trans-Sasakian manifolds of type $(0,0), (\alpha, 0)$ and $(0, \beta)$ are cosympletic [3], α-Sasakian [4] and β-Kenmotsu manifold [4,5] respectively. Trans-Sasakian manifolds have been studied in [6,7] and by many others.

On the other hand, contact conformal curvature tensor field was introduced and defined by Jeong et al. [8] in a $(2n+1)$-dimensional Sasakian manifold which was constructed from the conformal curvature tensor field defined by Kitahara et al. [9] in a Kaehler manifold by using the Boothby-Wang's fibration. Contact conformal curvature tensor has also been studied in [10] and [11].

2. Preliminaries

Let M be a $(2n+1)$-dimensional almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g), where φ is a (1, 1) tensor field, ξ is a vector field, η-is a 1-form and g is a compatible Riemannian metric such that [3]

(2.1) $$\varphi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \varphi\xi = 0,\ \eta(\varphi X) = 0,$$

(2.2) $$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

(2.3) $$g(\varphi X, Y) = -g(X, \varphi Y),\ g(X, \xi) = \eta(X),$$

for all $X, Y \in TM$. The fundamental 2-form Φ of the almost contact metric structure (φ, ξ, η, g) is defined as

(2.4) $$\Phi(X, Y) = g(X, \varphi Y) = -g(\varphi X, Y),$$

since φ is a skew-symmetric with respect to g.

An almost contact metric manifold M is called trans-Sasakian manifold if [2]

(2.5) $$(\nabla_X \varphi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\},$$

where ∇ is Levi-Civita connection of Riemannian metric g and α, β are smooth functions on M. From (2.5) it follows that

(2.6) $$\nabla_X \xi = -\alpha\varphi X + \beta\{X - \eta(X)\xi\},$$

(2.7) $$(\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y).$$

In a $(2n+1)$-dimensional trans-Sasakian manifold M, the following relations hold [6]

(2.8) $$\begin{aligned} R(X, Y)\xi = {} & (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - (X\alpha)\varphi Y - (X\beta)\varphi^2(Y) \\ & + 2\alpha\beta[\eta(Y)\varphi X - \eta(X)\varphi Y] + (Y\alpha)\varphi X + (Y\beta)\varphi^2(X), \end{aligned}$$

(2.9) $$\begin{aligned} R(\xi, X)Y = {} & (\alpha^2 - \beta^2)[g(X, Y)\xi - \eta(Y)X] + 2\alpha\beta[g(\varphi Y, X)\xi \\ & - \eta(Y)\varphi X] + (Y\alpha)\varphi X + g(\varphi Y, X)(grad\,\alpha) \\ & + (Y\beta)[X - \eta(X)\xi] - g(\varphi X, \varphi Y)(grad\,\beta), \end{aligned}$$

(2.10) $$2\alpha\beta+(\xi\alpha)=0,$$

(2.11) $$S(X,\xi)=[2n(\alpha^2-\beta^2)-(\xi\beta)]\eta(X)-((\varphi X)\alpha)-(2n-1)(X\beta),$$

(2.12) $$\eta(R(X,Y)Z)=-g(R(X,Y)\xi,Z),$$

(2.13) $$\eta(R(X,Y)\xi)=\eta(R(\xi,X)\xi)=\eta(R(X,\xi)\xi)=0,$$

(2.14) $$\eta(R(\xi,X)Y)=(\alpha^2-\beta^2-(\xi\beta))g(\varphi X,\varphi Y).$$

In a $(2n+1)$-dimensional trans-Sasakian manifold if we put $\varphi(grad\alpha)=(2n-1)grad\beta,$ then we have

(2.15) $$(\xi\beta)=0,$$

(2.16) $$S(X,\xi)=2n(\alpha^2-\beta^2)\eta(X),$$

(2.17) $$\eta(R(\xi,X)Y)=(\alpha^2-\beta^2)g(\varphi X,\varphi Y),$$

(2.18) $$R(\xi,X)\xi=(\alpha^2-\beta^2)\{\eta(X)\xi-X\},$$

(2.19) $$R(X,\xi)\xi=-R(\xi,X)\xi.$$

Throughout the paper we consider the trans-Sasakian manifold under the condition $\varphi(grad\alpha)=(2n-1)grad\beta.$

In a $(2n+1)$-dimensional trans-Sasakian manifold the contact conformal curvature tensor field C_0 of type (1, 3) which is defined by [8] can be written as

(2.20)
$$\begin{aligned}C_0(X,Y)Z=&R(X,Y)Z+\frac{1}{2n}\{S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX\\&-g(X,Z)QY+S(X,Z)\eta(Y)\xi-S(Y,Z)\eta(X)\xi\\&+\eta(X)\eta(Z)QY-\eta(Y)\eta(Z)QX+S(\varphi X,Z)\varphi Y\\&-S(\varphi Y,Z)\varphi X+g(X,\varphi Z)Q(\varphi Y)-g(Y,\varphi Z)Q(\varphi X)\\&+2g(X,\varphi Y)Q(\varphi Z)+2S(\varphi X,Y)\varphi Z\}\\&+\frac{1}{2n(n+1)}\{2n^2-n-2+\frac{(n+2)r}{2n}\}\{g(Y,\varphi Z)\varphi X\\&-g(X,\varphi Z)\varphi Y-2g(X,\varphi Y)\varphi Z\}\\&+\frac{1}{2n(n+1)}\{n+2-\frac{(3n+2)r}{2n}\}\{g(Y,Z)X-g(X,Z)Y\}\\&-\frac{1}{2n(n+1)}\{4n^2+5n+2-\frac{(3n+2)r}{2n}\}\{\eta(Y)\eta(Z)X\\&-\eta(X)\eta(Z)Y+g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi\},\end{aligned}$$

where R,S,Q and r denote the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature respectively.

From (2.20), we also have

(2.21) $$C_0(X,Y)\xi = R(X,Y)\xi + (\alpha^2-\beta^2-2)\{\eta(Y)X-\eta(X)Y\},$$

(2.22) $$C_0(\xi,X)Y = R(\xi,X)Y + (\alpha^2-\beta^2-2)\{g(X,Y)\xi-\eta(Y)X\},$$

(2.23) $$\eta(C_0(X,Y)Z) = \eta(R(X,Y)Z) + (\alpha^2-\beta^2-2)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\},$$

(2.24) $$\eta(C_0(X,Y)\xi) = 0,$$

(2.25) $$\eta(C_0(\xi,X)Y) = 2(\alpha^2-\beta^2-1)\{g(X,Y)-\eta(X)\eta(Y)\}.$$

Definition. A $(2n+1)$-dimensional trans-Sasakian manifold M is said to be an η-Einstein manifold if its Ricci tensor S of type (0, 2) is of the form

(2.26) $$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a,b are smooth functions on M. If $b=0$, then the manifold M becomes an Einstein manifold.

3. Contact Conformally Flat Trans-Sasakian Manifold

Definition. A $(2n+1)$-dimensional trans-Sasakian manifold is said to be contact conformally flat if it satisfies the condition

(3.1) $$C_0(X,Y)Z = 0.$$

Now, we prove the following result:

Theorem 3.1. If a $(2n+1)$-dimensional trans-Sasakian manifold M is contact conformally flat, then $\alpha^2 = \beta^2 + 1$.

Proof. Let M be a $(2n+1)$-dimensional trans-Sasakian manifold. Suppose M is contact conformally flat then the condition $C_0(X,Y)Z=0$ holds. Now, using (3.1) in (2.20) and taking inner product on both sides by ξ, we get

(3.2) $$\begin{aligned}\eta(R(X,Y)Z) = \frac{1}{2n}[&g(X,Z)S(Y,\xi)-g(Y,Z)S(X,\xi)\\ &-\eta(X)\eta(Z)S(Y,\xi)+\eta(Y)\eta(Z)S(X,\xi)]\\ &+2[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)].\end{aligned}$$

In view of (2.12), (2.16) and (3.2), we get

(3.3) $$\begin{aligned}0 = 2(\alpha^2-\beta^2-1)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]+2\alpha\beta[g(\varphi Y,Z)\eta(X)\\ -g(\varphi X,Z)\eta(Y)]+(X\alpha)g(\varphi Y,Z)-(Y\alpha)g(\varphi X,Z)-(X\beta)g(\varphi Y,\varphi Z).\end{aligned}$$

Putting $X=\xi$ in (3.3) and using (2.1), (2.10) and (2.15), we obtain

(3.4) $$(\alpha^2-\beta^2-1)[g(Y,Z)-\eta(Y)\eta(Z)] = 0.$$

Since $g(Y,Z)-\eta(Y)\eta(Z)\neq 0,$ we have $(\alpha^2-\beta^2-1)=0.$ This implies that

(3.5) $$\alpha^2 = \beta^2 + 1.$$

This completes the proof of the theorem.

4. ξ-Contact Conformally Flat Trans-Sasakian Manifold

Definition. A trans-Sasakian manifold of dimension $(2n+1)$ is said to be ξ-contact conformally flat if the condition

(4.1) $$C_0(X,Y)\xi = 0$$

holds.

Theorem 4.1. Let M be a $(2n+1)$-dimensional trans-Sasakian manifold satisfying the condition $C_0(X,Y)\xi = 0,$ then $\alpha^2 = \beta^2 + 1.$

Proof. Let us consider a $(2n+1)$-dimensional trans-Sasakian manifold M which satisfies the condition $C_0(X,Y)\xi = 0.$ Then by virtue of (2.1), (2.3), (2.8), (2.16) and (4.1) in (2.20), we get

(4.2) $$\begin{aligned} 0 = 2(\alpha^2 - \beta^2 - 1)\{\eta(Y)X - \eta(X)Y\} - (X\alpha)\varphi Y + (X\beta)Y \\ -(X\beta)\eta(Y)\xi + 2\alpha\beta[\eta(Y)\varphi X - \eta(X)\varphi Y] + (Y\alpha)\varphi X \\ -(Y\beta)X + (Y\beta)\eta(X)\xi. \end{aligned}$$

Putting $X = \xi$ in (4.2) and using (2.1), (2.10) and (2.15), we obtain

(4.3) $$(\alpha^2 - \beta^2 - 1)\{\eta(Y)\xi - Y\} = 0.$$

Since $\eta(Y)\xi - Y = \varphi^2(Y) \neq 0,$ we have $(\alpha^2 - \beta^2 - 1) = 0.$ This yields

(4.4) $$\alpha^2 = \beta^2 + 1.$$

Thus the theorem is proved.

From theorem 3.1 and theorem 4.1, we can state the following result:

Theorem 4.2. Trans-Sasakian manifolds of dimension $(2n+1)$ which satisfy the conditions $C_0(X,Y)Z = 0$ and $C_0(X,Y)\xi = 0$ are equivalent.

5. Trans-Sasakian Manifold Satisfying $C_0(\xi, X).S = 0$

Consider a trans-Sasakian manifold M of dimension $(2n+1)$. Let S be the Ricci tensor of type (0, 2). We prove the following result:

Theorem 5.1. Let M be a $(2n+1)$-dimensional trans-Sasakian manifold. If M satisfies the condition $C_0(\xi, X).S = 0,$ then it is an Einstein manifold with scalar curvature $r = 2n(2n+1)(\alpha^2 - \beta^2).$

Proof. Let M be a $(2n+1)$-dimensional trans-Sasakian manifold which satisfies the condition

(5.1) $$C_0(\xi, X).S(U,V) = 0.$$

This condition implies that

(5.2) $$S\left(C_0(\xi, X)U, V\right) + S\left(U, C_0(\xi, X)V\right) = 0.$$

Putting $V = \xi$ in (5.2) and using (2.16) and (2.22), we obtain

(5.3) $$S(X,U) = 2n\left(\alpha^2 - \beta^2\right)g(X,U).$$

Taking an orthonormal frame field at any point of the manifold and contracting over X and U in (5.3), we get

(5.4) $$r = 2n(2n+1)\left(\alpha^2 - \beta^2\right).$$

From (5.3) and (5.4) it follows that the manifold M is an Einstein manifold with scalar curvature $r = 2n(2n+1)\left(\alpha^2 - \beta^2\right)$. This completes the proof of the result.

6. Trans-Sasakian Manifold Satisfying $C_0(\xi, X).C_0 = 0$

Let M be a $(2n+1)$-dimensional trans-Sasakian manifold. Suppose the condition $\left(C_0(\xi, X).C_0\right)(U,V)Z = 0$ holds in M. Then we have

Theorem 6.1. A $(2n+1)$-dimensional trans-Sasakian manifold satisfying the condition $C_0(\xi, X).C_0 = 0$ is contact conformally semi-symmetric if

$$0 = \left(\alpha^2 - \beta^2\right)\left\{2g(V,Z)X - g(X,V)Z - g(X,Z)V\right\}$$
$$-g\left(R(\xi,V)Z, X\right)\xi - R(X,V)Z.$$

Proof. Let us consider a $(2n+1)$-dimensional trans-Sasakian manifold which satisfies the condition $C_0(\xi, X).C_0(U,V)Z = 0,$ then by definition we have

(6.1)
$$0 = C_0(\xi, X)C_0(U,V)Z - C_0\left(C_0(\xi, X)U, V\right)Z$$
$$-C_0\left(U, C_0(\xi, X)V\right)Z - C_0(U,V)C_0(\xi, X)Z.$$

Using (2.22) in (6.1) we get

(6.2)
$$0 = R(\xi, X).C_0(U,V)Z + \left(\alpha^2 - \beta^2 - 2\right)[g\left(X, C_0(U,V)Z\right)\xi$$
$$-\eta\left(C_0(U,V)Z\right)X - g(X,U)C_0(\xi, V)Z + \eta(U)C_0(X,V)Z$$
$$-g(X,V)C_0(U,\xi)Z + \eta(V)C_0(U,X)Z - g(X,Z)C_0(U,V)\xi$$
$$+\eta(Z)C_0(U,V)X].$$

Taking inner product on both sides of (6.2) by ξ and using (2.24), we obtain

(6.3)
$$\begin{aligned}0 &= g\left(R(\xi,X).C_0(U,V)Z,\xi\right)+\left(\alpha^2-\beta^2-2\right)[g\left(X,C_o(U,V)Z\right)\\ &-\eta(X)\eta\left(C_0(U,V)Z\right)-g(X,U)\eta\left(C_0(\xi,V)Z\right)\\ &+\eta(U)\eta\left(C_0(X,V)Z\right)-g(X,V)\eta\left(C_0(U,\xi)Z\right)\\ &+\eta(V)\eta\left(C_0(U,X)Z\right)-\eta(Z)\eta\left(C_0(U,V)X\right)].\end{aligned}$$

Putting $U=\xi$ in (6.3) and using (2.22), (2.23) and (2.25), we get

(6.4)
$$\begin{aligned}0 &= g\left(R(\xi,X).C_0(\xi,V)Z,\xi\right)\\ &+\left(\alpha^2-\beta^2-2\right)[g\left(R(\xi,V)Z,X\right)+\eta\left(R(X,V)Z\right)\\ &+\left(\alpha^2-\beta^2\right)\{\eta(Z)g(X,V)-2\eta(X)g(V,Z)+\eta(V)g(X,Z)\}].\end{aligned}$$

This implies that

(6.5)
$$\begin{aligned}&R(\xi,X).C_0(\xi,V)Z\\ &=\left(\alpha^2-\beta^2-2\right)[\left(\alpha^2-\beta^2\right)\{2g(V,Z)X-g(X,V)Z\\ &-g(X,Z)V\}-g\left(R(\xi,V)Z,X\right)\xi-R(X,V)Z].\end{aligned}$$

From this it follows that the manifold is contact conformally semi-symmetric if the right hand side of (6.5) vanishes, i. e., if

$$\begin{aligned}0 &= \left(\alpha^2-\beta^2\right)\left\{2g(V,Z)X-g(X,V)Z-g(X,Z)V\right\}\\ &-g\left(R(\xi,V)Z,X\right)\xi-R(X,V)Z.\end{aligned}$$

This completes the proof of the theorem.

7. An Example of a 3-dimensional Trans-Sasakian Manifold

Let us consider a 3-dimensional manifold $M=\left\{(x,y,z):(x,y,z)\in\mathbb{R}^3\right\}, z\neq 0,$ where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields

(7.1)
$$e_1=e^z\left(\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}\right),\ e_2=e^z\frac{\partial}{\partial y},\ \ e_3=\frac{\partial}{\partial z},$$

which are linearly independent at each point of M. Now we define a semi-Riemannian metric g on M as

(7.2)
$$g(e_1,e_3)=g(e_2,e_3)=g(e_1,e_2)=0,$$

(7.3)
$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1.$$

Let η be a 1-form defined by $\eta(Z)=g(Z,e_3)$ for any vector field $Z\in M$ and φ be a (1, 1) tensor field defined by

(7.4) $$\varphi(e_1)=e_2,\ \varphi(e_2)=-e_1,\ \varphi(e_3)=0.$$

The linearity property of φ and g yields that

(7.5) $$\eta(e_3)=1,\ \varphi^2(Z)=-Z+\eta(Z)e_3,\ g(\varphi Z,\varphi U)=g(Z,U)-\eta(Z)\eta(U)$$

for any $Z,U\in M$.

If we take $e_3=\xi$ in (7.5), (φ,ξ,η,g) defines an almost contact metric structure on M.

By the definition of Lie bracket and (7.1) we have

$$[e_1,e_2]=e_1e_2-e_2e_1=e^z\left(\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}\right)e^z\frac{\partial}{\partial y}-e^z\frac{\partial}{\partial y}\left(e^z\frac{\partial}{\partial x}+ye^z\frac{\partial}{\partial z}\right)$$
$$=ye^ze_2-e^{2z}e_3.$$

Proceeding same way we obtain $[e_2,e_3]=-e_2$ and $[e_1,e_3]=-e_1$. Thus we have

(7.6) $$[e_1,e_2]=ye^ze_2-e^{2z}e_3,\ [e_2,e_3]=-e_2,\ [e_1,e_3]=-e_1.$$

Let ∇ be the Levi-Civita connection with respect to g then we have the Koszul's formula

(7.7) $$\begin{aligned}2g(\nabla_XY,Z)&=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)\\&+g([X,Y],Z)-g([Y,Z],X)+g([Z,X],Y).\end{aligned}$$

By the use of (7.2), (7.3) and (7.6), (7.7) yields

(7.8) $$\begin{cases}\nabla_{e_1}e_3=-e_1+\frac{1}{2}e^{2z}e_2,\ \nabla_{e_1}e_2=\frac{1}{2}e^{2z}e_3,\ \nabla_{e_1}e_1=e_3,\\ \nabla_{e_2}e_3=-e_2-\frac{1}{2}e^{2z}e_1,\ \nabla_{e_2}e_2=e_3+ye^ze_1,\ \nabla_{e_2}e_1=-ye^ze_2+\frac{1}{2}e^{2z}e_3,\\ \nabla_{e_3}e_3=0,\ \nabla_{e_3}e_2=-\frac{1}{2}e^{2z}e_1,\ \nabla_{e_3}e_1=\frac{1}{2}e^{2z}e_2,\end{cases}$$

In view of (2.6), (7.2), (7.3) and (7.4) we have

$\nabla_{e_1}\xi=\beta e_1-\alpha e_2,\ \nabla_{e_2}\xi=\alpha e_1+\beta e_2$ and $\nabla_{e_3}\xi=0$ for $e_3=\xi$. Comparing these equations with (7.8) (first column), we get $\alpha=-\frac{1}{2}e^{2z}$ and $\beta=-1$.

Again, by virtue of (2.7) and $(\nabla_X\eta)Y=\nabla_X\eta(Y)-\eta(\nabla_XY)$ we obtain

$$\left(\nabla_{e_1}\eta\right)e_1=\beta=-1,\ \left(\nabla_{e_2}\eta\right)e_1=\alpha=-\frac{1}{2}e^{2z},\ \left(\nabla_{e_3}\eta\right)e_1==0.$$

Thus from above calculation the conditions (2.6) and (2.7) are satisfied and the structure (φ,ξ,η,g) is a trans-Sasakian structure of type (α,β) where $\alpha=-\frac{1}{2}e^{2z}$ and $\beta=-1$. Consequently $M^3(\varphi,\xi,\eta,g)$ is a trans-Sasakian manifold.

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