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On some nonlinear fractional PDEs in physics

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Abstract

In this paper, we applied relatively new fractional complex transform (FCT) to convert the given fractional partial differential equations (FPDEs) into corresponding partial differential equations (PDEs) and Variational Iteration Method (VIM) is to find approximate solution of time- fractional Fornberg-Whitham and time-fractional Wu-Zhang equations. The results so obtained are re-stated by making use of inverse transformation which yields it in terms of original variables. It is observed that the proposed algorithm is highly efficient and appropriate for fractional PDEs arising in mathematical physics and hence can be extended to other problems of diversified nonlinear nature. Numerical results coupled with graphical representations explicitly reveal the complete reliability and efficiency of the proposed algorithm.

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Keywords: Jumarie's fractional derivatve; Fractional Fornberg-Whitham equation; fractional Wu-Zhang equations; variational iteration method; fractional complex transform.

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1. Introduction

The nonlinear partial differential equations (NPDEs) are encountered in various disciplines, such as physics, mechanics, chemistry, biology, mathematics and engineering. Nonlinear partial differential equations [1-29] are of extreme importance. Recently, scientists have observed that number of real time problems is modeled by fractional nonlinear differential equations [4-5,7,10,19,21-25] which are very hard to tackle. Transform is an important method to solve mathematical problems. Recently the fractional complex transform [20-23] was suggested to convert fractional order differential equations with modified Riemann-Liouville derivatives [24-25] into integer order differential equations, and the resultant equations can be solved by different methods.

This paper is devoted to the study of time-fractional Fornberg-Whitham equation, modified time fractional Fornberg-Whitham equation [6-12], time-fractional Wu-Zhang equation [24, 25]. The Fornberg–Whitham equation was first proposed for studying the qualitative behavior of wave breaking. The time fractional Fornberg–Whitham equation can be written as

$$D_t^{\alpha}u - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_xu_{xx}\,,\ t > 0\,, 0 < \alpha \leq 1, \tag{1}$$

Subject to the initial conditions

$$u(x,0) = f(x),$$

and modifying the nonlinear term uu_x in Eq. (1) by u^2u_x, He et al. proposed in [12] the modified time fractional Fornberg-Whitham equation

$$D_t^{\alpha}u - u_{xxt} + u_x = uu_{xxx} - u^2u_x + 3u_xu_{xx}\,,\ t > 0\,, 0 < \alpha \leq 1, \tag{2}$$

and time-fractional Wu-Zhang equation

$$\begin{aligned} &D_t^{\alpha}u + uu_x + vu_y + w_x = 0\ , \\ &D_t^{\alpha}v + uv_x + vv_y + w_y = 0,\ 0 < \alpha \leq 1 \\ &D_t^{\alpha}w + (uw)_x + (vw)_y + \tfrac{1}{3}\left(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy}\right) = 0 \end{aligned} \tag{3}$$

with initial conditions

$$\begin{aligned} &u(x,y,0) = f_1(x,y), \\ &v(x,y,0) = f_2(x,y), \\ &w(x,y,0) = f_3(x,y). \end{aligned}$$

where w is the elevation of the water, u is the surface velocity of water along x -direction, and v is the surface velocity of water along y-direction. Wu and Zhang derived three sets of model equations for modeling nonlinear and dispersive long gravity waves travelling in two horizontal directions on shallow waters of uniform depth. Eq. (3) is one of these equations, Wu-Zhang equation (which describes (2+1)-dimensional dispersive long wave). The fractional derivatives are considered in the Jumarie sense. The basic motivation of this paper is the extension of a very reliable and efficient technique namely Variational Iteration Method using Complex Transform (VIMCT) to find approximate solutions of time-fractional Fornberg-Whitham and system of time-fractional Wu-Zhang equations. The convergence of the proposed variational iteration method using fractional derivative is addressed in [28-29]. It is observed that the proposed algorithms is fully synchronized with the complexity of fractional differential equations, Numerical results coupled with graphical representations explicitly reveal the complete reliability and efficiency of the proposed algorithm.

2. Definitions

Definition 2.1 Jumarie's fractional derivative [24-25] is a modified Riemann-Liouville derivative defied as

$$D_x^{\alpha}f(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_0^x (x-t)^{-\alpha-1}\big(f(t)-f(0)\big)dt, & \alpha < 0\ , \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha}\big(f(t)-f(0)\big)dt, & 0 < \alpha < 1, \\ \left(f^{\alpha-n}(x)\right)^n & n \leq \alpha < n+1, n \geq 1. \end{cases} \tag{4}$$

where $f\colon R \to R, x \to f(x)$ denotes a continous (but not necessarily differentiable) function. Some useful formulas and results of Jumarie's Modified Riemann-Liouville Derivatives are as follows:

$$D_x^{\alpha}c = 0\,, \alpha > 0, c = constant. \tag{5}$$

$$D_x^{\alpha}[c\,f(x)] = c\,D_x^{\alpha}f(x), \alpha > 0, c = cosntant. \tag{6}$$

$$D_x^{\alpha}\, x^{\beta} = \frac{\Gamma(\alpha+1)x^{\beta-\alpha}}{\Gamma(1+\beta-\alpha)}, \beta > \alpha > 0. \tag{7}$$

$$D_x^{\alpha}[f(x)g(x)] = [D_x^{\alpha} f(x)]\, g(x) + f(x)\, [D_x^{\alpha} g(x)]. \tag{8}$$

$$D_x^{\alpha} f\,\big(x(t)\big) = f_x^{`}(x).x^{\alpha}(t). \tag{9}$$

3. Variational Iteration Method (VIM) using Complex Transform

The nonlinear differential equations [13-16] can be expressed in the operator form as

$$D_t^{\alpha}(u(x,t)) + R(u(x,t)) + N(u(x,t)) = 0. \tag{10}$$

where D_t^{α} is the time-fractional Jumarie's fractional derivative , $N(u)$is the nonlinear operator and $R(u)$ is some linear operator.

The complex transform requires

$$S = t^{\alpha}$$

Using the basic properties of the fractional derivative [20], we can convert the fractional derivative into classical derivative.

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial u}{\partial T}\frac{\partial T^{\alpha}}{\partial t^{\alpha}} = \sigma \frac{\partial u}{\partial T}, \tag{11}$$

where σ, is defined [20], Eq. (10) becomes

$$\sigma\, u_S(\mathrm{u}(x,t)) + R(\mathrm{u}(x,t)) + N(\mathrm{u}(x,t)) = 0, \tag{12}$$

where $u_S = \frac{\partial u}{\partial S}$

$$L_S(u(x,t))) + R(u(x,t))) + N(u(x,t)))=0, \tag{13}$$

where L_S is the linear differential operator.

According to Variational Iteration Method, we construct a correction functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,\xi)[L_s u_n(\mathrm{x},\xi) + R\tilde{u}_n(\mathrm{x},\xi) + N\tilde{u}_n(x,t)]d\xi, \tag{14}$$

where λ is the general Lagrangian multiplier which can be indentified optimally by the variational theory ,the subscript n denotes the *nth* order approximation, and $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

Its stationary conditions can be obtained as follows:

$$\lambda^{/}(\mathrm{x},\xi) = 0,$$

$$1 + \lambda(\mathrm{x},\xi) = 0.$$

The Lagrange multiplier, therefore, can be obtained as $\lambda = -1$, and the following variational iteration formula can be obtained as

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t L_s u_n(x,\xi) + R(u_n) + N(u_n)d\xi. \tag{15}$$

Considering $u_0(x,0) = u(x,0)$, we can easily find the components of the iterative formula defined in (15).

Applying backward substitution to the computed components $S = t^{\alpha}$, we get

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{16}$$

4. Numerical Applications

Example 4.1 Consider the following time- fractional Fornberg-Whitham equation defined in Eq. (1)

$$D_t^\alpha u - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx}\,, \quad t > 0\,, 0 < \alpha \leq 1, \tag{17}$$

Subject to the initial conditions

$$u(x,0) = e^{\frac{x}{2}}.$$

Applying procedure defined in (11-15),

$$u_0 = e^{\frac{x}{2}},$$

$$u_1(x,S) = -\frac{1}{2\sigma} e^{\frac{x}{2}}(-2 + S),$$

$$u_2(x,S) = \frac{1}{8\sigma} e^{\frac{x}{2}}(8 - 5S + S^2),$$

$$u_3(x,S) = -\frac{1}{96\sigma} e^{\frac{x}{2}}(-96 + 63S - 18S^2 + 2S^3),$$

$$\vdots$$

applying backward substitution

$$u_1(x,t) = -\frac{1}{2\sigma} e^{\frac{x}{2}}(-2 + t^\alpha),$$

$$u_2(x,t) = \frac{1}{8\sigma} e^{\frac{x}{2}}(8 - 5t^\alpha + t^{2\alpha}),$$

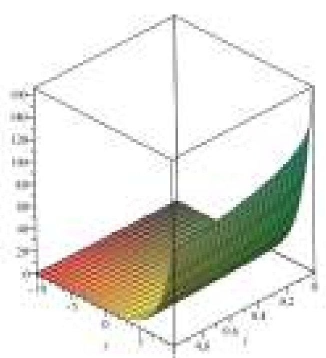
$$U_3(x,t) = -\frac{1}{96\sigma} e^{\frac{x}{2}}(-96 + 63t^\alpha - 18t^{2\alpha} + 2t^{3\alpha}), \tag{18}$$

$$\vdots$$

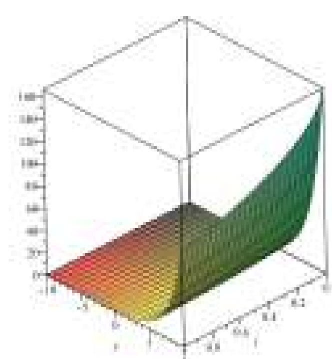
and so on. The exact solution of the time- fractional Fornberg-Whitham equation is obtained [6].

$$u(x,t) = e^{\left(\frac{x}{2} - \frac{2t}{3}\right)}. \tag{19}$$

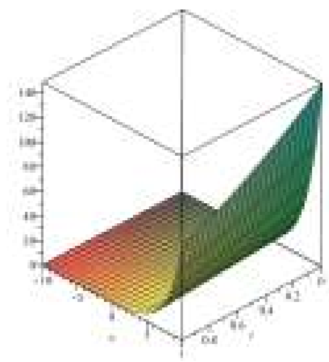
Graphical representation of exact solution (19) and the approximate solutions (18) for $\alpha = 0.3, 0.6, 1$.



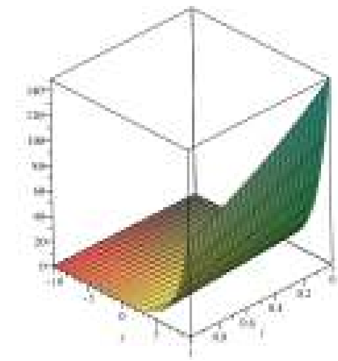
(a) $\alpha = 0.3$



(b) $\alpha = 0.6$



(c) $\alpha = 1$



(d) Exact Solution

Example 4.2 Consider the following time-fractional modified Fornberg-Whitham equation defined in Eq. (2)

$$D_t^{\alpha} u - u_{xxt} + u_x = u u_{xxx} - u^2 u_x + 3 u_x u_{xx}\,,\ t > 0\,, 0 < \alpha \leq 1, \tag{20}$$

Subject to the initial conditions

$$u(x,0) = a\,\mathrm{sech}^2(cx),$$

where $a = \frac{3}{4}\left(\sqrt{15} - 5\right), c = \frac{1}{20}(\sqrt{10(5 - \sqrt{15})})$.

Applying procedure defined in (11-15)

$$u_0 = a\,\mathrm{sech}^2(cx),$$

$$\begin{aligned} u_1(x,S) = \frac{a}{\sigma \cosh^7(cx)} &[\cosh^5(cx) 2c \sinh(cx)\cosh^4(cx) S \\ &+ 2a^2 c \sinh(cx)\, S 32 c^3 a \sinh(cx)\cosh^2(cx) + 60 a c^3 \sinh(cx)\, S, \end{aligned}$$

$$\vdots,$$

and so on. Applying backward substitution

$$u_0 = a\,\mathrm{sech}^2(cx),$$

$$u_1(x,t) = \frac{a}{\sigma \cosh^7(cx)} [\cosh^5(cx)\, 2c \sinh(cx)\cosh^4(cx) t^{\alpha} + 2a^2 c \sinh(cx)\, t^{\alpha} - 32 c^3 a \sinh(cx)\cosh^2(cx) + 60 a c^3 \sinh(cx)\, t^{\alpha}, \tag{21}$$

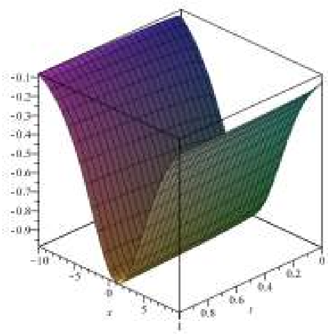
$$\vdots,$$

and so on.

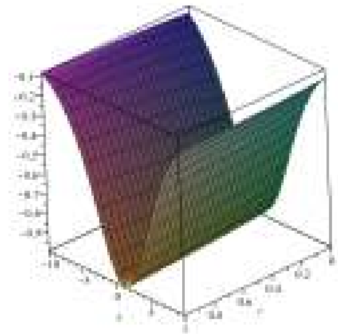
The exact solution of the time- fractional modified Fornberg-Whitham equation is obtained [6].

$$u(x,t) = a\mathrm{sech}^2\left(c\,\left(x - \left(5 - \sqrt{15}\right)t\right)\right). \tag{22}$$

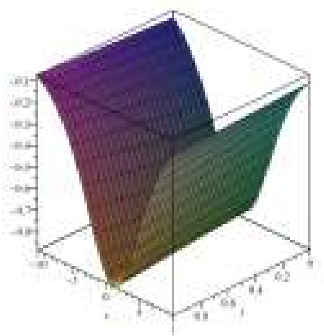
Graphical representation of exact solution (22) and the approximate solutions (21) for $\alpha = 0.3, 0.6, 1$.



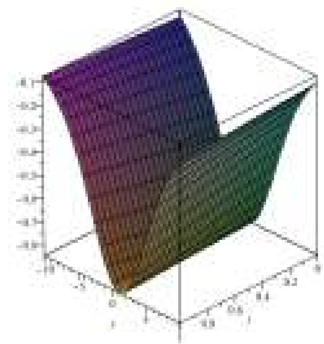
(a) $\alpha = 0.3$



(b) $\alpha = 0.6$



(c) $\alpha = 1$



(d) Exact Solution

Example 4.3 Consider time- fractional Wu-Zhang equation,

$$\begin{aligned}
&D_t^{\alpha}u + uu_x + vu_y + w_x = 0,\\
&D_t^{\alpha}v + uv_x + vv_y + w_y = 0,\\
&D_t^{\alpha}w + (uw)_x + (vw)_y + \tfrac{1}{3}\left(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy}\right) = 0,
\end{aligned} \tag{23}$$

with initial conditions

$$u(x, y, 0) = -\frac{k_3 + k_2 b_0}{k_1} + \frac{2\sqrt{3}}{3} k_1 \tanh(k_1 x + k_2 y),$$

$$v(x, y, 0) = b_0 + \frac{2\sqrt{3}}{3} k_2 \tanh(k_1 x + k_2 y),$$

$$w(x, y, 0) = \frac{2}{3}(k_1^2 + k_2^2)\,\mathrm{sech}^2(k_1 x + k_2 y),$$

where b_0, k_1, k_2 and k_3 are arbitrary constants. Applying the procedure defined above (10-15), we get

$$u_0(x, y, S) = -\frac{k_3 + k_2 b_0}{k_1} + \frac{2\sqrt{3}}{3} k_1 \tanh(k_1 x + k_2 y),$$

$$v_0(x, y, S) = b_0 + \frac{2\sqrt{3}}{3} k_2 \tanh(k_1 x + k_2 y),$$

$$u_0(x, y, S) = \frac{2}{3}(k_1^2 + k_2^2)\,\mathrm{sech}^2(k_1 x + k_2 y),$$

$u_1(x,y,S) = \frac{1}{3\,\sigma\cosh^2(k_1x+k_2y)}(-3k_3\cosh^2(k_1x+k_2y)^2 - 3k_2\,c\cosh^2(k_1x+k_2y)^2 + 2\sqrt{3}k_1^2\sinh(k_1x+k_2y)\cosh(k_1x+k_2y) + 2\sqrt{3}k_1^2Sk_3),$

$v_1(x,y,S) = \frac{1}{3\sigma\cosh^2(k_1x+k_2y)}(3c\cosh^2(k_1x+k_2y)^2 + 2\sqrt{3}\,k_2\sinh(k_1x+k_2y)\cosh(k_1x+k_2y) + 2\sqrt{3}k_2Sk_3),$

$w_1(x,y,S) = -\frac{2}{3\sigma\cosh^3(k_1x+k_2y)}\left(2\sinh(k_1x+k_2y)\,Sk_3k_1^2 + 2\sinh(k_1x+k_2y)\,Sk_3k_2^2 - k_1^2\cosh(k_1x+k_2y)\right),$

$\vdots,$

and so on.

Applying backward transformation

$$u_1(x,y,t) = \frac{1}{3\sigma\cosh^2(k_1x+k_2y)}(-3k_3\cosh^2(k_1x+k_2y)^2 - 3k_2\,c\cosh^2(k_1x+k_2y)^2 + 2\sqrt{3}k_1^2\sinh(k_1x+k_2y)\cosh(k_1x+k_2y) + 2\sqrt{3}k_1^2t^\alpha k_3), \quad (24)$$

$$v_1(x,y,t) = \frac{1}{3\sigma\cosh^2(k_1x+k_2y)}(3c\cosh^2(k_1x+k_2y)^2 + 2\sqrt{3}\,k_2\sinh(k_1x+k_2y)\cosh(k_1x+k_2y) + 2\sqrt{3}k_2t^\alpha k_3), \quad (25)$$

$$w_1(x,y,t) = -\frac{2}{3\sigma\cosh^3(k_1x+k_2y)}(2\sinh(k_1x+k_2y)\,t^\alpha k_3k_1^2 + 2\sinh(k_1x+k_2y)\,t^\alpha k_3k_2^2 - k_1^2\cosh(k_1x+k_2y)), \quad (26)$$

$\vdots,$

and so on. Finally, we have

$$u(x,y,t) = \lim_{n\to\infty} u_n(x,y,t),$$

$$v(x,y,t) = \lim_{n\to\infty} v_n(x,y,t),$$

$$w(x,y,t) = \lim_{n\to\infty} w_n(x,y,t).$$

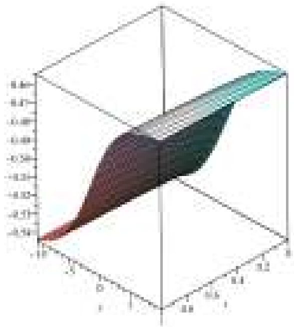
The exact solution of time-fractional Wu-Zhang Equations [26], is given by

$$u(x,y,t) = -\frac{k_3+k_2b_0}{k_1} + \frac{2\sqrt{3}}{3}k_1\tanh(k_1x+k_2y+k_3t). \quad (27)$$

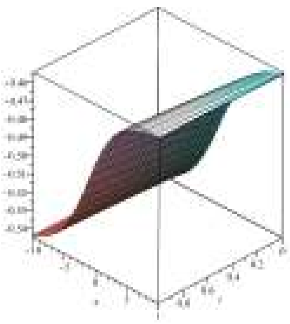
$$v(x,y,t) = b_0 + \frac{2\sqrt{3}}{3}k_2\tanh(k_1x+k_2y+k_3t). \quad (28)$$

$$w(x,y,t) = \frac{2}{3}\left(k_1^2+k_2^2\right)\operatorname{sech}^2(k_1x+k_2y+k_3t). \quad (29)$$

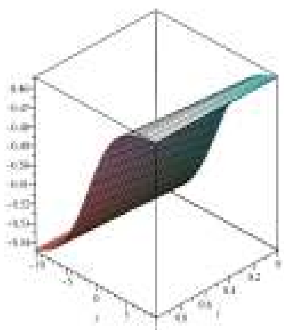
Graphical representation of exact solution (27) and the approximate solution (24) for $\alpha = 0.3, 0.6, 1$.



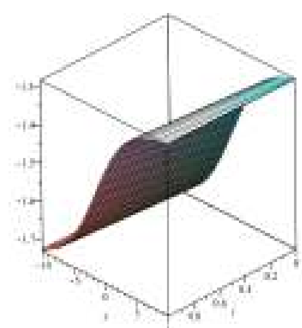
(a) $\alpha = 0.3$



(b) $\alpha = 0.6$

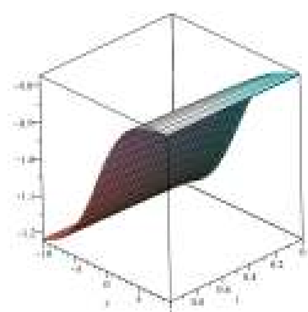


(c) $\alpha = 1$

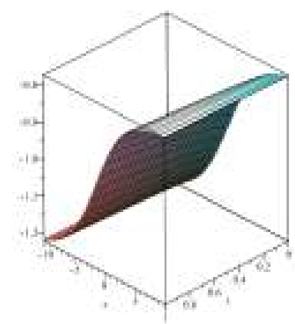


(d) Exact Solution

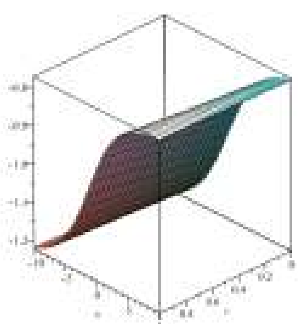
Graphical representation of the exact solution (28) and the approximate solution (25) for $\propto =$ *0.3,0.6,1*.



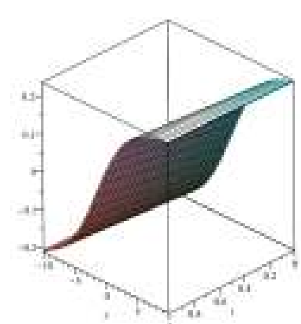
(a) $\alpha = 0.3$



(b) $\alpha = 0.6$

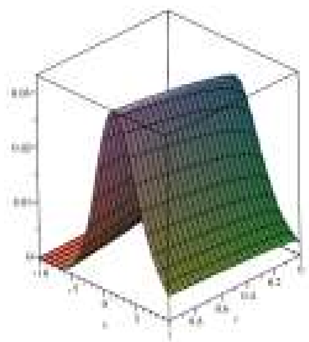


(c) $\alpha = 1$

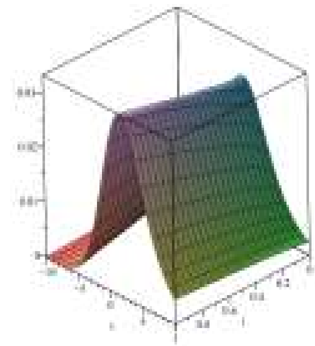


(d) Exact Solution

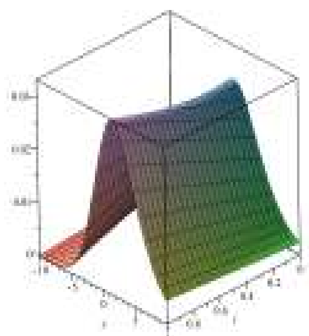
Graphical representation of exact solution (29) and the approximate solution (26) for $\propto = 0.3, 0.6, 1$.



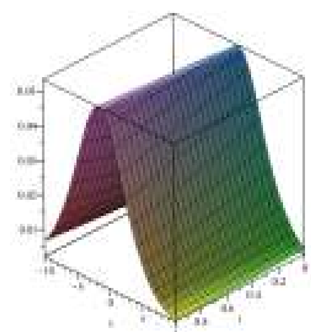
(a) $\alpha = 0.3$



(b) $\alpha = 0.6$



(c) $\alpha = 1$



(d) Exact Solution

5. Conclusions

Applied fractional complex transform (FCT) proved very effective to convert the given partial differential equations (PDEs) into corresponding partial differential equations (PDEs) and the same is true for its subsequent effect in Variational Iteration Method (VIM) which was implemented on the transformed PDEs. Computational work fully re-confirms the reliability and efficacy of the proposed algorithm and hence it may be concluded that presented scheme may be applied to a wide range of complex physical problems.

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