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CATEGORY OF FUZZY SETS

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Abstract

In the present paper, the concept of category theory, the category of fuzzy sets and the theory of flou sets are introduced. This article also shows the isomorphism between the category of fuzzy sets and flou sets.

Keywords

Category, flou sets, fuzzy sets, morphism, functor

Introduction

The application of algebra in geometry brings out certain strong analogies between the two subjects. The continuous mapping form one space into another corresponds to homomorphism of the associated groups. If the topological space is a surface, it may have a tangent plane, which has the structure of a vector space. Then a mapping from one surface to another which is a linear transformation of the corresponding tangent spaces. Such anologies led to a new subject of study known as category theory. Category theory is developed in 1942 in the work of Sounders MacLane(1909-1995) and Samuel Eileanberg(b.1913).

Category theory

In category theory the basic elements are sets of 'objects' and mapping among the objects called 'morphism' . The objects in category theory may be vector spaces and morphim may be linear transformation. Also the objects may be topological spaces and the morphism may be continuous function. All these particular objects are in compassed in the more general subjects of category theory(Cooke, 1997).

A category K is a domain of mathematical discourse comprising a collection of objects, for each pair (A, B) of objects a collection K (A, B) of morphisms:

$f : A \to B$ or $A \xrightarrow{f} B$ with domain A and codomain B, together with a law of composition:

$$g.f : A \to C$$

where $f : A \to B$ & $g : B \to C$,
which is associative and has identities
$id_A : A \to A$

A functor H from category K to category L sends objects A to objects AH and morphisms $f : A \to B$ in K to morphisms $fH : AH \to BH$ in L as

If $f = id_A : A \to A$ then

$fH = id_{AH} : AH \to AHg$ and

if $f = A \xrightarrow{g} B \xrightarrow{h} C$ then

$fH = AH \xrightarrow{gH} BH \xrightarrow{hH} CH.$

Category of fuzzy sets

The theory fuzzy sets extended the basic mathematical concepts of set. In view of the fact that set theory is the corner-stone of modern Mathematics a new and more general framework of mathematics was established. Fuzzy mathematics is just a kind of mathematics developed in theis framework and a fuzzy topology is just a kind of topology developed on fuzzy sets. Hence, fuzzy mathematics being such kind of mathematical theory which contains wider content than the classical theory inspired the author to work on fuzzy topological spaces (Zadeh, 1965).

Let SET (l) be the category of fuzzy sets whose objects are pairs (X, A) where X is a set and $A \in I^X$.

A morphism $(X, A) \xrightarrow{\Phi} (Y, B)$ is a function $\Phi : X \to Y$ such that $Bo\Phi : X \to I$ with $BO\Phi A$.

It is also possible to represent SET (l) with the category of Flou sets defined by :

Definition

A flou (vegue) set of a set X is a mapping $f : I \to P(X)$ provided

(i) f is non increasing with $f(0) = 1 \,\&\, f(1) = 0$.

(ii) $f(\underset{i\in\Delta}{Sup}\,\alpha_i) = \underset{i\in\Delta}{\cap} f(\alpha_i)$ for any $\underset{i\in\Delta}{(\alpha_i)} \subset l$.

Remarks

For a Flou set, $A \subset f(\lambda) \Rightarrow \exists \lambda^* > \lambda$ such that $A \subset f(\lambda^*)$

There is a complete isomorphism between I^X & $P(X)^I$.

Theorem

Modelling theorem : A mapping $t : P(X)^I \to I^X$ is bijective s.t. $f(\lambda)$ is the λ th cut of tf provided

$$t\,f = \begin{cases} \vee \{\lambda | x \in f(\lambda)\} \; for\; x \in f(0) \\ 0, \quad otherwise. \end{cases}$$

Proof : I. $X \in (f)_\lambda \Rightarrow f(X) > \lambda$
$\Rightarrow \exists \lambda^* > \lambda$

with $f(x) > \lambda^*$

$\Rightarrow x \in (f)\lambda^* \Rightarrow x \in f(\lambda^*) \subset f(\lambda) \Rightarrow (f)_\lambda \subset f(\lambda)$

Also, $x \in f(\lambda) \Rightarrow \exists \lambda^*$

with $x \in f(\lambda^*) \Rightarrow f(x) > \lambda^* > \lambda$
$\Rightarrow X \in (f)_\lambda$
$\Rightarrow f(\lambda) \subset (f)_\lambda$

II. $f_1 = f_2 \Rightarrow (f_1)_\lambda = (f_2)_\lambda$
$\Rightarrow f_1(\lambda) = f_2(\lambda)$
$\Rightarrow f_1 = f_2$

Remarks

For Lucasiewicz $W_3 = \{0,\ 1/2,\ 1\}$, Flou set can be identified with a pair: (A, B) where A, B are crisp subsets of X with $A \subseteq B$.

Theorem

The category SET (l) & Fl (l) are isomorphic.

Proof : Let the objects of SET (l) be pairs (X, A) whose morphisms $(X,A)\overset{\Phi}{\rightarrow}(Y,B)$ are functions $\Phi : X \rightarrow Y$ s.t. $B0\Phi : X \rightarrow I$ where $B0\Phi \geq A$. Let the category of vegue sets be Fl(l), the objects of which are the pairs (X, f) $(f \in p(X))$ whose morphisms $(X,f)\overset{\Psi}{\rightarrow}(Y,g)$ are functions $\Psi : X \rightarrow Y$, s.t $\Psi(f(\alpha)) \subseteq g(\alpha)$.

It is possible to build a pair of functors SET (l) $\underset{\text{Ø}}{\overset{\text{Ö}}{\rightleftarrows}}$ Fl(l) s.t. $\text{Ö}\,0\,\text{Ø} = id$, $\text{Ø}\,0\,\text{Ö} = id$.

For SET (l), $\text{Ö}(X,\ A) = (X,\ f_A)$, where

$$f_A(\alpha) = \{\mathbf{x} \in \mathbf{X} | A(\mathbf{x}) \geq \alpha\} \forall \alpha \in \mathbf{I} \ ,$$

and

$$\text{Ø}(X, f) = (X, A_f), \text{ where}$$

$$A_f(X) = \sup\{\alpha \in I | X \in f(a)\} \forall x \in X \quad ,$$

and

$$\text{Ö}(\Phi) = \Phi, \text{Ø}(\Psi) = \Psi .$$

It can easily be verified, using modelling theorem that the pair of funtors $(\text{Ö}\ \ \text{Ø})$ leads to categorical isomorphism.

Remarks

If L is a complete lattice and $\sum$ is a Heyting algebra then SET (l) can be generalized to SET (L) & SET $(\sum\ )$.

References

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