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RIEMANN INTEGRAL: SIGNIFICANCE AND LIMITATIONS

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Abstract

This article basically defines the Riemann integral starting with the definition of partition of a closed interval. It throws a light on the importance and necessity of the Riemann condition of integralbility of a function and explains how the concept of R- integral only on the basis of upper and lower integral is not always practical. It has been suggested that a better rout is to abandon the Reimann integral for Lebesgue integral in real analysis and other fields of mathematical science.

Key words

Partition; upper sum; lower sum; closed and bounded set; R-integral

Introduction

Integration arose in connection with the evaluation of areas of plane regions and thus amounted to finding out the limit of a sum when the number of terms tended to infinity, each term tending to zero. Realization that the subject could be looked up on inverse of differentiation came afterwards. The reference to integration from summation point of view was always associated with the geometrical concepts. The subject of integration is generally introduced as the inverse of differentiation at elementary stage. The basic relationship between the process of integration and differentiation of function is known as the fundamental theorem of integral calculus.

The German mathematician Riemann (1826-1866) in 1850 gave a purely arithmetic approach to formulate an independent theory of integration. He separated the concept of integration from its companion differentiation. The fundamental theorem of calculus became a result that held only for a restricted set of integrable functions. Riemannian theory led others to invent other integration theories, the most significant being Lebesgue theory of integration (Kishan, 2012).

To formulate an independent theory of integration, Riemann gave a purely arithmetic treatment to the subject and thus developed the subject entirely free from the intuitive dependence on geometrical concepts. Many refinements and generalizations of the subject followed, the most noteworthy being Lebesgue theory of integration. Riemann integral was the first definition of the integral of a function on an interval in the real analysis. While the Riemann integral is unsuitable for many theoretical purposes, it is one of the easiest integrals to define. Some of these technical deficiencies can be remedied by the Riemann-Stieltjes integral, and most of them disappeared in the Lebesgue integral. Riemann, no doubt, acquired his interest in problems connected with trigometric series through contact with Dirichlet when he spent a year in Berlin. He almost attended Dirichlet's lectures.

Riemann sum

In order to estimate an area, we need a partition of the interval [a, b]. A parition P of the closed interval [a,b] is defined as a finite set of points.

$P = \{x_0, x_1, x_2, \ldots\ldots\ldots\ldots x_n\}$ such that

$a = x_0 < x_1 < x_2 < \ldots\ldots\ldots\ldots\ldots\ldots < x_n = b$.

If *f* is a function defined on that interval [a, b], then the Riemann Sum of *f* with respect to the partition P is defined as :

$$R(f,P) = \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} f(t_k)(\Delta x_k)$$

Where t_k is an arbitrary number in the interval $[x_{k-1}, x_k]$

Upper and lower Riemann sum

There are two useful cases:

i) The use of supermum M_k of *f*(x) in the interval $[x_{k-1}, x_k]$, produces the upper sum.

$$U(f,P) = \sum_{k=1}^{n} M_k \,\Delta x_k$$

ii) The use of infimum m_k of *f* (x) in the interval [] produces the lower sum

$$L(f,P) = \sum_{k=1}^{n} m_k \,\Delta x_k$$

Size of Riemann sum

Let P be a partition of the closed interval[a,b] and *f*(x) be bounded function defined on the interval ,then the lower (upper) sum is increasing (decreasing) with respect to the refinement of partitions .i.e,

L(*f*,P') L(*f*,P) or U (*f*,P') U(*f*,P) for every refinement P' of the partition P.

Also for any partition P we have L(*f*,P) R (*f*,P) U (*f*,P)

That is, lower sum is always less than or equal to the upper sum (Apostal,1992).

Riemann integral

The upper and lower Riemann integrals are defined as :

$I^*(f)$ = inf { U (*f*,P) : P is a partition of [a,b]}

(*f*) = Sup. {L(*f*,P) : p is a partition of [a,b]}

Then if I^* (*f*) = (*f*) the function *f*(x) is called Riemann Integral (R-Integrable) and the Riemann integral of *f* (x) over the interval [a,b] is denoted by $\int_a^b f(x)dx$

It can be concluded by definition that U (*f*, P) and L(*f*, P) depend on the chosen partition, while the upper and lower integrals are independent of partitions. But this definition is not practical, since we need to find the sup and inf over any partition.

Example 1

Is *f*(x) = x^2 R – integrable on [a,b]?

It is complicated to prove that this function is integrable. We do not have a simple condition to tell us whether this, or any other function, is integrable.

But, we should be able to generalize the prooffor this particular example to a wider set of funtions.

First, note that in the definition of upper and lower integral it is not necessary to take the sup and inf over all partitions .If P is partition and P' is a refinement of P, then

L (*f*,p') L (*f*,P) and U (*f*,P') U(*f*,p)

Thus, partitions with large intervals (large norms,) do not contribute to the sup or inf. We look at partitions with a small norms.

Second we take any and a partition P with <ε/2. Then,

(),

Where is the sup of *f* over and d_j is the infimum over the interval.

Since *f*(x) is increasing over [0, 1], we know that the sup is achieved on the right side of each subinterval , the infimum on the left side. Then,

$|U\ (f,P) - L(f,P)| \leq \sum_{j=1}^{n} |c_{j-} d_j| \ (x_{j-}\ x_{j-1}),$

$=\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \ (x_j\text{-}x_{j-1})$

To estimate this sum, we use the Mean Value Theorem for *f*(x) = x² .

$|f(x) - f(y)| \leq |f'(c)||x - y|$ For c between x and y .

Since $|f'(c)| \leq 2$ for c $\in [0,1]$ $\Rightarrow |f(x) - f(y)| \leq 2\,|x - y|$

But P was chosen with $|P| < \varepsilon/2$

$\Rightarrow |f(x_j) - f(x_{j-1})| \leq 2\,|x_j - x_{j-1}| \leq 2\varepsilon/2 = \varepsilon$

Then, $|U\ (f,P) - L(f,P)| \leq \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \ (x_j\text{-}x_{j-1})$

$\leq \varepsilon \sum_{j=1}^{n} (x_j - x_{j-1}) = \varepsilon(1\text{-}0) = \varepsilon$

Since P was arbitrary but with small norm – sufficient for the upper and lower integral, the upper and lower integral must exist and be equal to one common limit L .

Let's calculate L. It is easy, since now we know that the function is integrable. Then, we take a suitable partition to find the value of the integral. For example, we take the following partition

x_j=j/n for j = 0,1,2,................, n .

Then, the upper sum:

$U\ (f, P) = \sum_{j=1}^{n} c_j\ (x_j\text{-}x_{j-1})$

$= \sum_{j=1}^{n} f(x_j) 1/n$

$= \sum_{j=1}^{n} (j/n)^2 1/n$

$= 1/n^3 \sum_{j=1}^{n} j^2$

$= 1/n^3\ [1/6\ n\ (n+1)(2n+1)] = 1/6\ (n+1)\ (2n+1)/n^2$

Since we know that the upper integral exists and is equal to L, the limit as n goes to infinity of the above expression must also converge to L. Then, L=1/3

Example 2

Is the Dirichlet function R- integrable?

The Drichlet function is defined as

f(**x**) = 1 if x belongs to Q

= 0 if x doesn't belong to Q

We have that U (*f*,P) = 1 and L (*f*, P) = 0, regardless of P . Then,. Thus, the Drichlet function is not R-integrable over the interval [a, b](Keyton,2014).

Conclusion

It is difficult to establish the integral of any given function, and not every function is Riemann integrable. The Riemann easier condition to check the Riemann integrability of functions is stated as follows:

A bounded function *f*(x) defined on the closed, bounded interval [a,b] is R- integrable if and only if for every o there exists at least one partition P_ε such that

$$|U\ (f, P) - \ L(f, P)| < \ \varepsilon$$

The above inequality holds for every partition P with small enough norm. (More finer than P_ε)

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