

## Some Continuous Properties on Fuzzy Metric Space

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### Abstract

This paper deals with the concept of fuzzy metric space and their basic topological properties with illustrative examples. We define continuous mappings from a fuzzy metric space  $(U, F, *)$  into a fuzzy metric space  $(V, F, *)$  and prove some properties of continuous mappings including the composition function of two fuzzy continuous functions to be a fuzzy continuous function and the Cartesian product of two fuzzy metric space is a fuzzy metric space. To clarify the result, one numerical example is also presented.

**Keywords:** Fuzzy metric space; continuous mapping; open ball; convergent.

### Introduction

In a Simply way, a metric space is a non-empty set  $U$  connected to a function  $d$  of two variables that allows us to calculate the distance between the points. In more complex mathematics, determining the distance between objects like sequences, sets, and functions is just as important as determining the distance between two integers and vectors. Numerous strategies exist in this area to find a suitable concept of a metric space. As a result, new notions of distance are used in convergence and continuity. Many renowned mathematicians have discussed a number of generalizations of metric spaces.

Zadeh (1965) developed fuzzy sets, which served as the foundation for fuzzy mathematics. The ambiguity in daily existence is symbolized by this structure. Later, other authors developed the concepts of fuzzy space by applying various properties of general topology to fuzzy sets. Kramosil and Michalek, (1975) first proposed the concept of fuzzy metric space. George and Veeramani (1994) revised the notion of fuzzy metric space with the introduction of convergent sequence and Cuachy sequence. Researchers have developed continuous properties in fuzzy metric space utilizing the concepts of compatible map, implicit relation, weakly compatible map, and  $R$  weakly compatible maps. Kider and Hussian (2014) introduced some properties of continuous and uniform continuous mappings in standard fuzzy metric spaces connecting with convergence and uniform convergences properties. Similarly, Haseeb (2017) by introducing some fixed-point results in fuzzy 2-metric space and fuzzy 3- metric space connected these results with continuity and uniform continuity. Bhandari et al. (2024) introduced some topological properties on the cartesian product in fuzzy b-metric space, using the idea that if any two fuzzy b-metric spaces are complete then their Cartesian product is also complete. Similarly, using the notion of convergent sequence and Cauchy sequence in fuzzy b-metric space some, common fixed-point theorems are proved (Bhandari, Manandhar & Jha (2025)). In this paper we introduce the continuous properties in fuzzy metric space and also discuss the convergence

and uniform convergence properties in complete fuzzy metric space. If a sequence of function defined from a topological space to a fuzzy metric space and if it is uniformly convergence to any function then this function is continuous. Some illustrative examples are provided to verify the results.

## 2. Preliminaries

**Definition 2.1.** Schweizer & Sklar (1960) A binary operation  $*$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is continuous triangular norm (t-norm) if for all  $p, q, r, s \in [0, 1]$ , the following conditions are satisfied

- (i)  $p*1=p$
- (ii)  $p*q=q*p$
- (iii) If  $p \leq r$  and  $q \leq s$  then  $p*q \leq r*s$
- (iv)  $p*(q*r) = (p*q) *r$
- (v)  $*$  is continuous.

$Tp(p,q)=p.q$ ,  $Tm(P, q) = \min(p, q)$  and  $TL(p, q) = \max(p+q-1, 0)$  are the examples of t-norms.

**Definition 2.2.** George & Veeramani (1994) A 3-tuple  $(U, F, *)$  is said to be a fuzzy metric space if  $U$  is an arbitrary set,  $*$  is a continuous t-norm and  $F$  is a set defined on  $U \times U \times F \times (0, \infty)$  satisfying the following conditions, for all  $u, v, w \in U$  and  $t, s > 0$

- (i)  $F(u, v, t) > 0$ ,
- (ii)  $F(u, v, t) = 1$  for all  $t > 0$  if and only if  $u = v$
- (ii)  $F(u, v, t) = F(v, u, t)$ ,
- (iv)  $F(u, v, t) * F(v, w, s) \leq F(u, w, t + s)$  for all  $t, s > 0$ ,
- (v)  $F(u, v, .): (0, \infty) \rightarrow [0, 1]$  is continuous

Where  $F(u, v, t)$  is the degree of nearness between  $u$  and  $v$  with respect to  $t > 0$ .

The above condition (ii) is also equivalent to  $F(u, v, t) < 1$  for all  $u, v$  and  $t > 0$ .

**Example 2.3.** Let  $U = \mathbb{R}$ , if  $(U, d)$  be a metric space, by denoting  $a * b = a .b$  for all  $a, b \in [0, 1]$  and  $F$  be a fuzzy set on  $U \times U \times [0, 1]$  defined as below:

$$F(u, v, t) = \left[ \exp\left(\frac{u-v}{t}\right) \right]^{-1} \text{ where } u, v \in \mathbb{R} \text{ and } t > 0. \text{ Then } (U, F, *) \text{ is a fuzzy metric space.}$$

**Note.** The above example can also be hold if we define the t-norm as  $p * q = \min\{p, q\}$ , so that  $F$  is a fuzzy metric with respect to any continuous t-norm.

**Definition 2.4.** George & Veeramani (1994) A sequence  $\{u_n\}$  in a fuzzy metric space  $(U, F, *)$  is said to be convergent to  $u \in U$  if  $\lim_{n \rightarrow \infty} F(u_n, u, t) = 1$  for all  $t > 0$ .

And is Cauchy sequence if there exists  $n_0 \in \mathbb{N}$  such that for all  $m, n \geq n_0$  we have

$$\lim_{n \rightarrow \infty} F(u_n, u_m, t) = 1.$$

A fuzzy metric space in which every Cauchy sequence is convergent is called complete.

**Example 2.5.** Let  $U = \mathbb{R}^+$ , with the metric  $d$  defined by  $d(u, v) = |u - v|$  and t- norm  $u*v = \min\{u, v\}$ , we defined as

$$F(u, v, t) = \frac{t}{t+d(u,v)} \text{ for all } u, v \in \mathbb{R}^+, t > 0. \text{ Then } (U, F, *) \text{ is a complete fuzzy metric.}$$

**Definition 2.6.** George & Veeramani (1994) A fuzzy metric space  $(U, F, *)$  is said to be compact if every sequence  $\{u_n\}$  in  $U$  has a convergence subsequence in  $U$ .

**Definition 2.7.** Let  $V$  be any nonempty set and  $(U, F, *)$  be a fuzzy metric space. Then a sequence  $\{f_n\}$  of functions from  $U$  to  $V$  is said to converge uniformly to a function  $f$  from  $U$  to  $V$ , if given  $r, t > 0$  with  $0 < r < 1$ , there exists  $n_0 \in \mathbb{N}$  such that

$$F(f_n(u), f(u), t) > 1-r$$

for all  $n \geq n_0$  and for all  $u \in U$ .

**Definition 2.8.** Gregori & Sapena (2002) A mapping  $F: U \times U \rightarrow [0, 1]$  is said to be continuous in the fuzzy metric space  $(U, F, *)$  iff whenever  $u_n \rightarrow u, v_n \rightarrow v$  then  $\lim_{n \rightarrow \infty} (u_n, v_n, t) = F(u, v, t)$  for each  $t > 0$ .

By using the self mappings the continuity of a function can be defined as:

Two self mappings  $f$  and  $g$  on a fuzzy metric space  $(U, F, *)$  are said to be continuous on  $U$  if

$$\lim_{n \rightarrow \infty} f g u_n = f u \text{ and } \lim_{n \rightarrow \infty} g f u_n = g u, \text{ whenever } \{u_n\} \text{ is a sequence in } U \text{ such that}$$

$$\lim_{n \rightarrow \infty} f u_n = \lim_{n \rightarrow \infty} g u_n = u, \text{ for some } u \in U.$$

**Definition 2.9.** Sedghi & Shobe (2006) The open ball  $B(u, r, t)$  with center  $u \in U$  and the radius  $r$  with  $u \in (0, 1), t > 0$  in a fuzzy metric space  $(U, F, *)$  is defined as

$$B(u, r, t) = \{v \in U: F(u, v, t) > 1-r\}.$$

**Definition 2.10.** Sedghi & Shobe (2006) If  $A$  be a non empty subset of a fuzzy metric space  $(U, F, *)$  then it is said to be open if there is a point  $a \in A$  and  $r \in (0, 1)$  with  $t > 0$  such that  $B(a, r, t) \subset A$ .

Similarly a subset  $B$  of a fuzzy metric space  $(U, F, *)$  is said to be closed if  $B^c$  is open.

### 3. Main Result

**Theorem 3.1.** Let  $(U_1, F_1, *)$  and  $(U_2, F_2, *)$  be any two fuzzy metric spaces, for  $(u_1, u_2), (v_1, v_2) \in U_1 \times U_2$  and  $t > 0$  define

$$F((u_1, u_2), (v_1, v_2), t) = F_1(u_1, v_1, t) * F_2(u_2, v_2, t).$$

Then  $F$  is a fuzzy metric on  $U_1 \times U_2$ .

**Proof.** (i) Since  $F_1(u_1, v_1, t) > 0$  and  $F_2(u_2, v_2, t) > 0$ , this implies that

$$F_1(u_1, v_1, t) F_2(u_2, v_2, t) > 0.$$

Therefore,

$$F((u_1, u_2), (v_1, v_2), t) > 0.$$

(ii) Suppose that for all  $t > 0, (u_1, u_2, t) = (v_1, v_2, t)$ . This implies that  $u_1 = v_1$  and  $u_2 = v_2$  for all  $t > 0$ .

Hence,

$$F_1(u_1, v_1, t) = 1$$

and

$$F_2(u_2, v_2, t) = 1.$$

It follows that

$$F(u, v, t) = 1,$$

where  $u = (u_1, u_2)$  and  $v = (v_1, v_2)$ .

Conversely, suppose that  $F(u, v, t) = 1$ , where  $u = (u_1, u_2)$  and  $v = (v_1, v_2)$ . This implies that

$$F_1(u_1, v_1, t) F_2(u_2, v_2, t) = 1.$$

Since

$$0 < F_1(u_1, v_1, t) < 1$$

and

$$0 < F_2(u_2, v_2, t) \leq 1,$$

it follows that

$$F_1(u_1, v_1, t) = 1$$

and

$$F_2(u_2, v_2, t) = 1.$$

Thus,  $u_1 = v_1$  and  $u_2 = v_2$ . Therefore,  $u = v$ .

(iii) To prove that  $F(u, v, t) = F(v, u, t)$ :

Observe that

$$F_1(u_1, v_1, t) = F_1(u_1, v_1, t)$$

and

$$F_2(u_2, v_2, t) = F_2(v_2, u_2, t).$$

It follows that for all  $(u_1, v_1), (v_1, v_2) \in U_1 \times U_2$  and  $t > 0$ ,

$$F((u_1, u_2), (v_1, v_2), t) = F((v_1, v_2), (u_1, u_2), t).$$

(iv) Since  $(U_1, F_1, *)$  and  $(U_2, F_2, *)$  are fuzzy metric spaces, we have that

$$F_1(u_1, w_1, t+s) \leq F_1(u_1, v_1, t) F_1(v_1, w_1, s)$$

and

$$F_2(u_2, w_2, t+s) \leq F_2(u_2, v_2, t) * F_2(v_2, w_2, s),$$

for all

$$(u_1, u_2), (v_1, v_2), (w_1, w_2) \text{ and } U_1 \times U_2$$

and  $s, t > 0$ . Therefore,

$$F((u_1, u_2), (w_1, w_2), t+s) = F_1(u_1, w_1, t+s) * F_2(u_2, w_2, t+s)$$

and

$$F((u_1, u_2), (w_1, w_2), t+s) \leq F_1(u_1, v_1, t) F_1(v_1, w_1, s) * F_2(u_2, v_2, t) * F_2(v_2, w_2, s)$$

$$< F_1(u_1, v_1, t) F_2(u_2, v_2, t) * F_1(v_1, w_1, s) * F_2(v_2, w_2, s)$$

$$< F((u_1, u_2), (v_1, v_2), t) * F((v_1, v_2), (w_1, w_2), s).$$

(v) Since  $F_1(u_1, v_1, t)$  and  $F_2(u_2, v_2, t)$  are continuous with respect to  $t$  and is continuous, it follows that

$$F((u_1, u_2), (v_1, v_2), t) = F_1(u_1, v_1, t) * F_2(u_2, v_2, t)$$

is also continuous.

Hence satisfying all the properties, we conclude that  $F$  is a fuzzy metric on  $U_1 \times U_2$ .

**Theorem 3.2.** *Let  $(U_1, F_1, *)$  and  $(U_2, F_2, *)$  be fuzzy metric spaces.*

*Define*

$$F((u_1, u_2), (v_1, v_2), t) = F_1(u_1, v_1, t) * F_2(u_2, v_2, t).$$

*Then  $F$  is a complete fuzzy metric on  $U_1 \times U_2$  if and only if  $(U_1, F_1, *)$  and  $(U_2, F_2, *)$  are complete.*

**Theorem 3.3.** *Let  $f_n: U \rightarrow V$  be a sequence of continuous functions from a topological space  $U$  to a fuzzy metric space  $(V, F, *)$ . If  $\{f_n\}$  converges uniformly to  $f$ , then  $f$  is continuous.*

**Proof.** Let  $U$  be the given topological space and  $(V, F, *)$  be the given fuzzy metric space.

For any open set  $A$  in  $V$ , let  $u_0 \in f^{-1}(A)$  and let  $v_0 = f(u_0)$ . Since  $A$  is open, we can find  $r, t > 0$  with  $0 < r < 1$  such that

$$B(v_0, r, t) \subseteq A.$$

Since  $r \in (0, 1)$ , we can find  $s \in (0, 1)$  such that

$$(1-s) * (1-s) * (1-s) > 1-r.$$

Since  $\{f_n\}$  converges to  $f$ , given  $s, t > 0$  with  $s \in (0, 1)$ , there exists  $n_0 \in \mathbb{N}$  such that

$$F\left(f_n(u), f(u), \frac{t}{s}\right) > 1 - s$$

for all  $n \geq n_0$ . Since, for all  $n \in \mathbb{N}$ ,  $\{f_n\}$  is continuous, we can find a neighborhood  $U$  of  $u_0$  for a fixed  $n \geq n_0$  such that

$$f_n(U) \subseteq B\left(f_n(u_0), s, \frac{t}{s}\right)$$

Hence

$$F\left(f_n(u), f(u), \frac{t}{3}\right) > 1 - s$$

for all  $u \in U$ . Now

$$F(f(u), f(u), t) \geq F\left(f(u), f_n(u), \frac{t}{3}\right) * F\left(f(u), f_n(u_0), \frac{t}{3}\right) * F\left(f_n(u), f_n(u_0), \frac{t}{3}\right)$$

and

$$F(f(u), f(u_0), t) \geq (1-s) * (1-s) * (1-s) > 1-r.$$

Thus,

$$f(u) \in B(f(u_0), r, t) \subseteq A$$

for all  $u \in U$ . Hence  $f(U) \subseteq A$ , and therefore  $f$  is continuous.

Now, we'll first provide a basic overview of continuous mapping in a fuzzy metric space.

Now we will verify the above example numerically.

**Example 3.5.** *Let  $V = \mathbb{R}$  with the standard fuzzy metric*

$$F(u, v, t) = \frac{t}{t + |u - v|}$$

*and the product t-norm  $(u * v) = u.v$ . This  $F$  is continuous in all variables and satisfies the fuzzy metric axioms. Let the domain be  $U = [0, 1]$  with the usual topology. Define*

$$f_n(u) = u + \frac{1}{n}, \quad u \in [0, 1]$$

Each  $\{f_n\}$  is continuous and the pointwise limit is

$$f(u) = \lim_{n \rightarrow \infty} f_n(u) = u,$$

which is also continuous (to be verified in the fuzzy sense below).

**For uniform convergence:** For all  $u \in [0, 1]$ ,

$$|f_n(u) - f(u)| = \frac{1}{n}$$

Hence

$$\sup |f_n(u) - f(u)| = \frac{1}{n} \rightarrow 0,$$

so  $f_n \rightarrow f$  uniformly. In the fuzzy metric, given  $\varepsilon \in (0, 1)$  and  $t > 0$ , choose  $N$  such that for all  $n \geq N$  and all  $u \in [0, 1]$ ,

$$F(f_n(u), f(u), t) = \frac{t}{t + |f_n(u) - f(u)|} = \frac{t}{t + \frac{1}{n}} > 1 - \varepsilon$$

Equivalently, we need

$$\frac{1}{n} \leq \frac{\varepsilon t}{1 - \varepsilon}$$

**Numerically:** Take  $\varepsilon = 0.10$  and  $t = 0.5$ . Then

$$\frac{\varepsilon t}{1 - \varepsilon} = \frac{0.05}{0.9} \approx 0.0556.$$

Choose  $n \geq 18$ . For any  $u \in [0, 1]$ ,

$$F(f_{18}(u), f(u), 0.5) = \frac{0.5}{0.5 + \frac{1}{18}} = \frac{0.5}{0.5555} \approx 0.9000\bar{0} \geq 0.9 = 1 - \varepsilon.$$

Thus, the convergence is uniform in the fuzzy metric.

**Continuity of the limit in the fuzzy metric space**

Continuity at  $u_0 \in [0, 1]$  means: given  $\varepsilon \in (0, 1)$  and  $t > 0$ , there exists  $\delta > 0$  such that if  $|u - u_0| < \delta$ , then

$$F(f(u), f(u_0), t) = \frac{t}{t + |u - u_0|} > 1 - \varepsilon.$$

This holds if  $|u - u_0| < \frac{\varepsilon t}{1 - \varepsilon}$ . Thus we may choose

Let  $u_0 = 0.30$ ,  $\varepsilon = 0.10$ ,  $t = 0.5$ . Then

$$\delta = \frac{0.10 \times 0.5}{1 - 0.10} = \frac{0.05}{0.9} \approx .0556.$$

If  $|u - 0.30| < 0.0556$ , then

$$F(f(u), f(0.30), 0.5) = \frac{0.5}{0.5 + |u - 0.30|} > \frac{0.5}{0.5 + 0.0556} \approx 0.9 = 1 - \varepsilon.$$

Hence  $f$  is continuous at  $u_0$ . Since  $u_0$  was arbitrary,  $f$  is continuous on  $[0, 1]$ .

**Lemma 3.4.** If  $B(u, r_1, t)$  and  $B(u, r_2, t)$  are any two open balls having same center  $u \in U$  and for  $t > 0$  with  $r_1 \in (0, 1)$ ,  $t > 0$  and  $r_2 \in (0, 1)$ ,  $t > 0$ , then we have either

$$B(u, r_1, t) \subseteq B(u, r_2, t) \text{ OR } B(u, r_2, t) \subseteq B(u, r_1, t)$$

**Lemma 3.5.** Let  $(U, F, *)$  be a fuzzy metric space. If  $\lim_{n \rightarrow \infty} u_n = u$  and  $\lim_{n \rightarrow \infty} v_n = v$  then

$$\lim_{n \rightarrow \infty} F(u_n, v_n, t) = F(u, v, t).$$

**Theorem 3.6.** Let  $(U, F, *)$  and  $(V, G, *)$  be any two fuzzy metric spaces and assume that  $A \subseteq U$  and the function  $f: A \rightarrow V$  is continuous at a point  $a \in A$ . If the sequence  $\{a_n\}$  in  $A$  converges to the point  $a$ , then the sequence  $\{f(a_n)\}$  converges to  $f(a)$ .

**Proof.** Let us assume that the function  $f: A \rightarrow V$  is continuous at the point  $a \in A$  and  $\{a_n\}$  be a sequence in  $A$ . Then we will show that the function  $\{f(a_n)\}$  is converges to  $f(a)$ . Let  $\varepsilon \in (0, 1)$  then there exists  $\delta \in (0, 1)$  such that for  $b \in A$   $F(a, b, t) > 1 - \delta$  implies  $a \in A$  such that  $F(f(a), f(b), t) > 1 - \varepsilon$  for  $t > 0$ .

As  $\lim_{n \rightarrow \infty} a_n = a$ , there exists a positive integer  $N$  such that  $n \geq N$  gives  $F(a_n, a, t) > 1 - \delta$ .

Which gives  $F(f(a_n), f(a), t) > 1 - \varepsilon$ .

Hence  $\lim_{n \rightarrow \infty} f(a_n) = f(a)$ .

Conversely let us assume that for every sequence  $\{a_n\}$  which converges to  $a$  having the property  $\lim_{n \rightarrow \infty} f(a_n) = f(a)$ . Then we will show that  $f$  is continuous at  $a$ .

On the contrary, let us assume that  $f$  is not continuous at  $a$ .

Then there must exist  $\varepsilon \in (0, 1)$  for which there is no  $\delta$  with the property  $\delta \in (0, 1)$  that can satisfy the condition  $b \in A$ ,  $F(b, a, t) > 1 - \delta$  which implies  $a \in A$  such that  $F(f(b), f(a), t) > 1 - \varepsilon$  for  $t > 0$ . Now for every  $n \in \mathbb{N}$  the number  $\frac{1}{n}$  is positive and hence there exists  $a_n \in A$  such that  $F(a_n, a, t) > 1 - \frac{1}{n}$ , but  $F(f(a_n), f(a), t) \leq 1 - \varepsilon$ . Which gives that the sequence  $\{a_n\}$  converges to  $a$  but the sequence  $\{f(a_n)\}$  does not converges to  $f(a)$ .

That leads to the contradiction. Hence  $f$  must be continuous.

**Theorem 3.7.** Let  $(A, F, *)$ ,  $(B, F, *)$  and  $(C, F, *)$  be the fuzzy metric spaces and let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be continuous mappings. Then the composition  $g \circ f$  is a continuous map of  $A$  into  $C$ .

**Proof.** Let  $E$  be open subset of  $C$ . By theorem 3.6,  $g(E)$  is an open subset of  $B$  and another application of the same theorem shows that  $f(g(E))$  is an open subset of  $A$ . Since  $(g \circ f)(E) = f(g(E))$ , it follows from the same theorem again that  $g \circ f$  is continuous.

**Theorem 3.8.** Let  $(U, F, *)$  and  $(V, F, *)$  be the fuzzy metric spaces and let  $f: U \rightarrow V$  be a mapping then the following statements are equivalent.

(i)  $f$  is continuous on  $U$ .

(ii)  $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$  for all subsets  $B$  of  $V$ .

(iii)  $f(A) \subseteq \overline{f(A)}$  for all subsets  $A$  of  $U$ .

**Proof.** (i)  $\Rightarrow$  (ii) Let  $B$  be a subset of  $V$ . Since  $\overline{B}$  is a closed subset of  $V$ ,  $f^{-1}(\overline{B})$  is closed in  $U$ .

Moreover  $f^{-1}(B) \subseteq f^{-1}(\overline{B})$ , and so  $f^{-1}(B) \subseteq \overline{f^{-1}(B)}$ . Where  $f^{-1}(B)$  is the smallest closed set containing  $f^{-1}(B)$ .

(ii)  $\Rightarrow$  (iii) Let  $A$  be a subset of  $U$ . Then if  $B = f(A)$ , we have  $A \subseteq f^{-1}(B)$  and  $A \subseteq f^{-1}(B) \subseteq \overline{f^{-1}(B)}$ .

Thus  $f(A) \subseteq \overline{f(f(A))} = \overline{f(A)}$ . In this work, we explored the concepts of continuity, convergence, and uniform convergence in fuzzy metric spaces.

We showed that if a sequence of functions from a topological space to a fuzzy metric space converges uniformly, the limit function remains continuous.

Recent developments, such as results on fuzzy b-metric spaces and their Cartesian products, further strengthen the structure of these spaces.

The study highlights how fuzzy metric spaces generalize classical metric ideas to handle uncertainty. Examples provided confirm the validity of the main theorems.

These findings create new opportunities for research in fixed point theory and topology.

Future work may focus on extending these results to higher-order fuzzy metric structures.

(iii)  $\Rightarrow$ (i) Let  $F$  be a closed set in  $V$  and set  $f^{-1}(F) = F$ . We need to show that  $F$  is closed in  $U$ , since  $F = F$ . This completes the proof.

### Conclusion

In this work, we explored the concepts of continuity using the notions of convergence and uniform convergence in fuzzy metric spaces. We showed that if a sequence of functions from a topological space to a fuzzy metric space converges uniformly, the limit function remains continuous. Recent developments, such as results on fuzzy b-metric spaces and their Cartesian products, further strengthen the structure of these spaces. Examples provided confirm the validity of the main theorem. These findings create new opportunities for research in fixed point theory and topology. Future work may focus on extending these results to higher-order fuzzy metric structures.

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