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On Locally Convex Topological Vector Space Valued Paranormed Function Space $(\ell_\infty(X, Y, \Phi, \xi, w, L), H_U)$ Defined by Orlicz Function

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Abstract

The aim of this paper is to introduce and study a new class $(\ell_\infty(X, Y, \Phi, \xi, w, L), H_U)$ of locally convex space *Y*- valued functions using Orlicz function Φ as a generalization of some of the well known sequence spaces and function spaces. Besides the investigation pertaining to the linear topological structures of the class $(\ell_\infty(X, Y, \Phi, \xi, w, L), H_U)$ when topologized it with suitable natural paranorm , our primarily interest is to explore the conditions pertaining the containment relation of the class $\ell_\infty(X, Y, \Phi, \xi, w)$ in terms of different ξ and *w* so that such a class of functions is contained in or equal to another class of similar nature.

Key words: locally convex topological vector space, seminorm, paranormed space, orlicz function, orlicz sequence space

Introduction

We begin with recalling some notations and basic definitions that are used in this paper.

A topological linear space *X* is a vector space *X* over a topological field **K** (most often the complex numbers C with their standard topologies) which is endowed with a topology $\Im$ such that if $x, y \in X$, $\alpha \in$ **K**; the mappings

(i) vector addition $X \times X \to X$ such that $(x, y) \to x + y$; and

(ii) scalar multiplication $\mathbf{K} \times X \to X$ such that $(\alpha, x) \to \alpha x$ are continuous.

This topology $\Im$ is called a *vector topology* or a *linear topology* on *X*. If $\Im$ is given by some metric then the topological vector space is called a linear metric space. All normed spaces or inner product spaces endowed with the topology defined by its norm or inner product are well-known examples of topological vector spaces.

A local base of topological vector space *X* is a collection B of neighbourhood θ such that every neighbourhood of θ contains the member of B.

A set *S* in a topological vector space *X* is said to be absorbing if for every $x \in X$ there exists an $\alpha > 0$ such that $x \in \nu S$ for all $\nu \in$ **C** such that $|\nu| \geq \alpha$;and *balanced* if $\nu S \subset S$ for every $\nu \in$ **C** such that $|\nu| \leq 1$.It is called *convex* in *X* if for every $\alpha \geq 0$, we have $\alpha S + (1-\alpha) S \subset S$; and absolutely convex in *X* if it is both balanced and convex.

Let *X* be a topological vector space with topology $\Im$.Then *X* is called *locally convex* if there exists a local base B whose members are convex.It is called locally bounded if θ has bounded neighbourhood, and separated (or Hausdorff) topological vector space if the underlying topology $\Im$ is separated.

In a locally convex topological vector space *X* there exists a fundamental system N of neighbourhoods of θ such that each $V \in N$ is absorbing, balanced and convex and for every $V \in N$ there exists $U \in N$ such that $U + U \subset V$.

The *gauge* or Minkowski functional (of a set *A* in a vector space *X* is a map $x \to q_A(x)$ from *X* into the extended set $R_+ \cup \{\infty\}$ of non-negative real numbers defined as follows:

$$q_A(x) = \begin{cases} \inf r, & \text{if there exists } r > 0 \text{ such that } x \in rA \text{ and} \\ \infty, & \text{if } x \notin rA \text{ for all } r > 0. \end{cases}$$

A seminorm (pseudonorm) on a linear space X over the scalar C with zero element θ is a subadditive function $p : X \rightarrow R_+$ satisfying

$$p(\alpha x) = |\alpha|\, p(x), \text{ for all } \alpha \in C \text{ and } x \in X.$$

Clearly if $p(x) = 0$ implies $x = \theta$, then p is a norm. In a vector space X the gauge of an absorbing and convex set is a seminorm.

Applying above theorem it can easily be proved that a locally convex topology $\mathfrak{I}$ of a locally convex topological vector space X can always be defined by a family of seminorms, infact by the family $\{p_i : i \in I\}$ of all seminorms which are continuous for the topology $\mathfrak{I}$.

Conversely, if X is a vector space equipped with a family $\{p_i : i \in I\}$ of seminorms then there exists a unique locally convex topology $\mathfrak{I}$ on X such that each p_i is $\mathfrak{I}$-continuous, (see , Rudin, 1991 and Park ,2005).

A paranormed space (S, H) is a linear space S with zero element θ together with a function $H : S \rightarrow R_+$ (called a paranorm on S) which satisfies the following axioms:

PN_1: $H(\theta) = 0$;

PN_2: $H(s) = H(-s)$ for all $s \in S$;

PN_3: $H(s_1 + s_2) \leq H(s_1) + H(s_2)$ for all $s_1, s_2 \in S$;

and

PN_4: Scalar multiplication is continuous.

Note that the continuity of scalar multiplication is equivalent to

(i) if $H(s_n) \rightarrow 0$ and $\alpha_n \rightarrow \alpha$ as $n \rightarrow \infty$, then $H(\alpha_n s_n) \rightarrow 0$ as $n \rightarrow \infty$

and

(ii) if $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$ and s be any element in S, then $H(\alpha_n s) \rightarrow 0$, see Wilansky (1978).

The concept of paranormed space is closely related to linear metric space, see Wilansky (1978) and its studies on sequence spaces were initiated by Maddox (1969) and many others. Parasar and Choudhary (1994), Bhardwaj and Bala (2007), Khan (2008), Basariv and Altundag (2009) and many others further studied various types of paranormed sequence spaces .

An Orlicz function is a function $\Phi : [0,\infty) \rightarrow [0,\infty)$ which is continuous, non decreasing and convex with

(i) $\Phi(0) = 0$,

(ii) $\Phi(t) > 0$ for $t > 0$, and

(iii) $\Phi(t) \rightarrow \infty$ as $t \rightarrow \infty$.

An Orlicz function Φ can be represented in the following integral form

$$\Phi(t) = \int_0^t q(x)\, dx$$

where q, known as the kernel of Φ, is right-differentiable for $x \geq 0$, $q(0) = 0$, $q(x) > 0$ for $x > 0$, q is non decreasing, and $q(x) \rightarrow \infty$ as $x \rightarrow \infty$ (see, Krasnosel'skiĭ and Rutickiĭ,1961).

Let X be a normed space over C, the field of complex numbers. Let $\omega(X)$ denotes the linear space of all sequences $\bar{x} = (x_k)$ with $x_k \in X$, $k \geq 1$ with usual coordinate wise operations.We shall denote ω (C) by ω.

Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to construct the sequence space $\ell_\Phi(X)$ of scalars (x_k) such that

$$\ell_\Phi(X) = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} \Phi\left(\frac{|x_k|}{r}\right) < \infty \text{ for some } r > 0 \right\}.$$

The space $\ell_\Phi(X)$ with the norm

$$\|x\|_\Phi = \inf\left\{ r > 0 : \sum_{k=1}^{\infty} \Phi\left(\frac{|x_k|}{r}\right) \leq 1 \right\}$$

becomes a Banach space which is called an *Orlicz sequence space*.The space $\ell_\Phi(X)$ is closely related to the space ℓ_p which is an Orlicz sequence space with $\Phi(x) = x^p : 1 \leq p < \infty$.

Subsequently, Kamthan and Gupta (1981), Rao and Ren (1991), Parashar and Choudhary (1994), Chen (1996) , Ghosh and Srivastava (1999), Rao and Subremanina (2004), Savas and Patterson (2005) ,Bhardwaj and Bala (2007),Khan (2008), Kolk (2011), Srivastava and Pahari (2011) , and many others have been introduced and studied the algebraic and topological properties of various sequence spaces using Orlicz function as a generalization of several well known sequence spaces.

The classes $\ell_\infty(X, Y, \Phi, \xi, w)$ and $\ell_\infty(X, Y, \Phi, \xi, w, L)$ of locally convex space valued functions

Let X be an arbitrary non empty set (not necessari countable) and $\mathcal{F}(X)$ be the collection of all fini subsets of X. Let $(Y, \mathfrak{I})$ be a Hausdorff locally conve topological vector space (lcTVS) over the field of comple numbers C and Y^* be the topological dual of Y. Let (Y) denotes the fundamental system of balanced, conve and observing neighbourhoods of zero vector θ of Y. will denote gauge or Minkowski functional of $U \in \mathcal{U}($ Thus,

$$D = \{p_U : U \in \mathcal{U}(Y)\}$$

is the collection of all continuous seminorms generati the topology $\mathfrak{I}$ of Y.

Let w and ν be any functions on $X \to R^+$, the set of positive real numbers, and

$\ell_\infty(X, R^+) = \{ w : X \to R^+$ such that $\sup_x w(x) < \infty\}$.
Further, we write ξ, μ for functions on $X \to C \setminus \{0\}$, and the collection of all such functions will be denoted by $s(X, C \setminus \{0\})$.

Now for $w \in \ell_\infty(X, R^+)$ and $L = \max\{1, \sup_x w(x)\}$.
We now introduce the following new class of lc TVS - Y valued functions:

$\ell_\infty(X, Y, \Phi, \xi, w) = \{\phi : X \to Y$: for each $p_U \in D$ and for some $r > 0$,

$$\sup_{x \in X} \Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{w(x)}}{r}\right) < \infty\}. \quad \ldots(1)$$

Further when $\xi : X \to C \setminus \{0\}$ is a function such that $\xi(x) = 1$ for all x,
then $\ell_\infty(X, Y, \Phi, \xi, w)$ will be denoted by $\ell_\infty(X, Y, \Phi, w)$ and when $w : X \to R^+$ is a function such that $w(x) = 1$ for all x,
then $\ell_\infty(X, Y, \Phi, \xi, w)$ will be denoted by $\ell_\infty(X, Y, \Phi, \xi)$.

Besides studying the class (1), we also deal the followir class of lc TVS - Y valued functions

$\ell_\infty(X, Y, \Phi, \xi, w, L) = \{\phi : X \to Y$: for each $p_U \in D$ and for some $r > 0$,

$$\sup_{x \in X} \Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{w(x)/L}}{r}\right) < \infty\}. \quad \ldots(2)$$

Actually, these classes are the generalizations of the familiar sequence and function spaces, studied in Srivastava *et al.* (1996), Srivastava (1996), Tiwari *et al.* (2008, 2010), Pahari (2011), Srivastava and Pahari (2011) using norm .

Main Results

In this section, we explore the conditions in terms of different w and ξ so that a class $\ell_\infty(X, Y, \Phi, \xi, w)$ of lc TVS Y - valued functions is contained in or equal to another similar class and thereby derive the conditions of their equality. Beside this, we shall also investigate some results that characterize the linear topological structures of $\ell_\infty(X, Y, \Phi, \xi, w, L)$ by endowing it with suitable natural paranorm H_U.

We shall denote the zero element of this space by θ by which we shall mean the function $\theta : X \to Y$ such that $\theta(x) = \theta$, for all $x \in X$.

Moreover,we shall frequently use the notations
$L = \sup_x w(x)$ and for scalar α, $A[\alpha] = \max(1, |\alpha|)$.

But when the functions $w(x)$ and $\nu(x)$ occur, then to distinguish L we use the notations $L(w)$ and $L(\nu)$ respectively.

Theorem 1: If $\nu : X \to R^+$, $w \in \ell_\infty(X, R^+)$ and $\xi \in s(X, C \setminus \{0\})$, then
$\ell_\infty(X, Y, \Phi, \xi, w) \subset \ell_\infty(X, Y, \Phi, \xi, \nu)$
if and only if $\limsup_x \dfrac{\nu(x)}{w(x)} < \infty$.

Proof:
For the sufficiency, assume that

$$\limsup_x \frac{\nu(x)}{w(x)} < \infty .$$

Then there exists a constant $d > 0$ such that
$\nu(x) < d\, w(x)$
for all but finitely many $x \in X$.

Now, if $\phi \in \ell_\infty(X, Y, \Phi, \xi, w)$, $r > 0$ is associated with ϕ and $p_U \in D$, then we have

$$\sup_{x \in X} \Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{w(x)}}{r}\right) < \infty .$$

This shows that there exists some positive real number η satisfying

$$\Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{w(x)}}{r}\right) \le \Phi\left(\frac{\eta}{r}\right),$$

for all but finitely many $x \in X$.
Since Φ is non decreasing, therefore
$[p_U(\xi(x)\,\phi(x))]^{w(x)} < \eta$
Since $\nu(x) < d\, w(x)$ and so if
$[p_U(\xi(x)\,\phi(x))] \le 1$,
then obviously
$[p_U(\xi(x)\,\phi(x))]^{\nu(x)} \le 1$;
and on the other hand if
$[p_U(\xi(x)\,\phi(x))] > 1$,
then

$$[p_U(\xi(x)\,\phi(x))]^{\nu(x)} < [p_U(\xi(x)\,\phi(x))]^{d\,w(x)} < \eta^d.$$

Therefore

$$[p_U(\xi(x)\,\phi(x))]^{\nu(x)} \le \max(1, \eta^d),$$

for all but finitely many $x \in X$.

This shows that for all but finitely many $x \in X$,

$$\Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{\nu(x)}}{r}\right) \le \Phi\left(\frac{\max(1, \eta^d)}{r}\right),$$

and therefore

$$\sup_{x \in X} \Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{\nu(x)}}{r}\right) < \infty .$$

Since $p_U \in D$ is an arbitrary, it shows that $\phi \in \ell_\infty(X, Y, \Phi, \xi, \nu)$ and hence
$\ell_\infty(X, Y, \Phi, \xi, w) \subset \ell_\infty(X, Y, \Phi, \xi, \nu)$.

Conversely, assume that
$\ell_\infty(X, Y, \Phi, \xi, w) \subset \ell_\infty(X, Y, \Phi, \xi, \nu)$
but

$$\limsup_x \frac{\nu(x)}{w(x)} = \infty.$$

Then there exists a sequence (x_k) of distinct points in X

such that for each $k \geq 1$,

$$\nu(x_k) > k\, w(x_k). \qquad \ldots (1)$$

Now, taking $y \in Y$ and $p_V \in D$ with $p_V(y) = 1$.
We define

$$\phi : X \to Y \text{ by}$$

$$x_k = \begin{cases} (\xi(x_k))^{-1}\, 2^{1/w(x_k)}\, y, & \text{for } x = x_k,\ k \geq 1, \text{ and} \\ \theta, & \text{otherwise}. \end{cases} \qquad \ldots (2)$$

Let $r > 0$ .Then for each $p_U \in D$,we have

$$\sup_{x \in X} \Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{w(x)}}{r} \right)$$
$$= \sup_{k \geq 1} \Phi\left(\frac{[p_U(\xi(x_k)\,\phi(x_k))]^{w(x_k)}}{r} \right)$$
$$= \sup_{k \geq 1} \Phi\left(\frac{[p_U(2^{1/w(x_k)}\, y)]^{w(x_k)}}{r} \right)$$
$$= \sup_{k \geq 1} \Phi\left(\frac{2[p_U(y)]^{w(x_k)}}{r} \right)$$
$$\leq \Phi\left(\frac{2A\, [(p_U(y))^{L(w)}]}{r} \right).$$

This shows that $\phi \in \ell_\infty(X, Y, \Phi, \xi, w)$. But in view of (1) and (2) ,we have

$$\sup_{x \in X} \Phi\left(\frac{[p_V(\xi(x)\,\phi(x))]^{\nu(x)}}{r} \right)$$
$$= \sup_{k \geq 1} \Phi\left(\frac{2^{\nu(x_k)/w(x_k)} . [p_V(y)]^{\nu(x_k)}}{r} \right)$$
$$\geq \sup_{k \geq 1} \Phi\left(\frac{2^k}{r} \right)$$
$$= \infty$$

and hence $\phi \notin \ell_\infty(X, Y, \Phi, \xi, \nu)$, a contradiction.

This completes the proof.

Theorem 2: If $w : X \to R^+$, $\nu \in \ell_\infty(X, R^+)$ and $\xi \in s(X, C \setminus \{0\})$, then
$$\ell_\infty(X, Y, \Phi, \xi, \nu) \subset \ell_\infty(X, Y, \Phi, \xi, w)$$
if and only if $\liminf_x \dfrac{\nu(x)}{w(x)} > 0$.

Proof:
For the sufficiency, suppose that

$$\liminf_x \frac{\nu(x)}{w(x)} > 0 .$$

Then there exists $m > 0$ such that
$$\nu(x) > m\, w(x)$$
for all but finitely many $x \in X$.

Let $\phi \in \ell_\infty(X, Y, \Phi, \xi, \nu)$, $r > 0$ is associated with ϕ and $p_U \in D$. Then we have

$$\sup_{x \in X} \Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{\nu(x)}}{r} \right) < \infty .$$

This shows that we can find some positive real number η satisfying

$$\Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{\nu(x)}}{r} \right) < \Phi\left(\frac{\eta}{r} \right),$$

for all but finitely many $x \in X$.
Since Φ is non decreasing, we have

$$[p_U(\xi(x)\,\phi(x))]^{\nu(x)} \leq \eta.$$

Since $\nu(x) > m\, w(x)$ and so if

$$[p_U(\xi(x)\,\phi(x))] \geq 1,$$

then

$$[p_U(\xi(x)\,\phi(x))]^{w(x)} \leq [p_U(\xi(x)\,\phi(x))]^{\nu(x)/m} \leq \eta^{1/m};$$

and on the other hand if

$$[p_U(\xi(x)\,\phi(x))] < 1,$$

then obviously

$$[p_U(\xi(x)\,\phi(x))]^{w(x)} < 1.$$

Therefore

$$[p_U(\xi(x)\,\phi(x)]^{w(x)} \leq \max(1, \eta^{1/m}),$$

for all but finitely many $x \in X$.

This shows that for all but finitely many $x \in X$,

$$\Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{w(x)}}{r} \right) \leq \Phi\left(\frac{\max(1, \eta^{1/m})}{r} \right),$$

and therefore

$$\sup_{x \in X} \Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{w(x)}}{r} \right) < \infty.$$

Since $p_U \in D$ is arbitrary, it follows that $\phi \in \ell_\infty(X, Y, \Phi, \xi, w)$ and hence

$$\ell_\infty(X, Y, \Phi, \xi, \nu) \subset \ell_\infty(X, Y, \Phi, \xi, w).$$

Conversely, assume that the inclusion holds but

$$\liminf_x \frac{\nu(x)}{w(x)} = 0 .$$

Then there exists a sequence (x_k) of distinct points in X such that for $k \geq 1$,

$$k\, \nu(x_k) < w(x_k). \qquad \ldots (3)$$

Now, taking $y \in Y$ and $p_V \in D$ with $p_V(y) = 1$, define $\phi : X \to Y$ by

$$x_k = \begin{cases} (\xi(x_k))^{-1}\, 2^{1/\nu(x_k)}\, y, & \text{for } x = x_k,\ k \geq 1, \text{ and} \\ \theta, & \text{otherwise}. \end{cases} \qquad \ldots (4)$$

Let $r > 0$. Then for each $p_U \in D$, we have

$$\sup_{x \in X} \Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{\nu(x)}}{r} \right)$$
$$= \sup_{k \geq 1} \Phi\left(\frac{[p_U(\xi(x_k)\,\phi(x_k))]^{\nu(x_k)}}{r} \right)$$
$$= \sup_{k \geq 1} \Phi\left(\frac{2\,[p_U(y)]^{\nu(x_k)}}{r} \right)$$
$$\leq \Phi\left(\frac{2A\, [(p_U(y))^{L(\nu)}]}{r} \right).$$

This implies that $\phi \in \ell_\infty(X, Y, \Phi, \xi, v)$.
But on the other hand, in view of (5) and (6), we get

$$\sup_{x\in X} \Phi\left(\frac{[p_V(\xi(x)\phi(x))]^{w(x)}}{r}\right)$$
$$= \sup_{k\geq 1} \Phi\left(\frac{[p_V(\xi(x_k)\phi(x_k))]^{w(x_k)}}{r}\right)$$
$$= \sup_{k\geq 1} \Phi\left(\frac{2^{w(x_k)/v(x_k)}\,[(p_V(y))]^{w(x_k)}}{r}\right)$$
$$\geq \sup_{k\geq 1} \Phi\left(\frac{2^k}{r}\right) = \infty,$$

shows that $\phi \notin \ell_\infty(X, Y, \Phi, \xi, w)$, a contradiction.

This completes the proof.

After combining Theorem 1 and Theorem 2, we get the following theorem:

Theorem 3: If $w, v \in \ell_\infty(X, \boldsymbol{R}^+)$ and $\xi \in s(X, \boldsymbol{C} \setminus \{0\})$, then

$$\ell_\infty(X, Y, \Phi, \xi, w) = \ell_\infty(X, Y, \Phi, \xi, v)$$
if and only if
$$0 < \lim\inf_x \frac{v(x)}{w(x)} \leq \lim\sup_x \frac{v(x)}{w(x)} < \infty.$$

Theorem 4: If $w \in \ell_\infty(X, \boldsymbol{R}^+)$ and $\xi \in s(X, \boldsymbol{C} \setminus \{0\})$, then

(i) $\ell_\infty(X, Y, \Phi, \xi) \subset \ell_\infty(X, Y, \Phi, \xi, w)$ if and only if
$$\lim\sup_x w(x) < \infty;$$
(ii) $\ell_\infty(X, Y, \Phi, \xi, w) \subset \ell_\infty(X, Y, \Phi, \xi)$ if and only if
$$\lim\inf_x w(x) > 0; \text{ and}$$
(iii) $\ell_\infty(X, Y, \Phi, \xi, w) = \ell_\infty(X, Y, \Phi, \xi)$ if and only if
$$0 < \lim\inf_x w(x) \leq \lim\sup_x w(x) < \infty.$$

Proof:

If we consider $w : X \to \boldsymbol{R}^+$ such that
$$w(x) = 1 \text{ for all } x \in X$$
and v is replaced by w in Theorems 1,2 and 3, we can easily prove the assertions (i), (ii) and (iii) respectively.

Theorem 5: If $w \in \ell_\infty(X, \boldsymbol{R}^+)$, then for any $\xi, \mu \in s(X, \boldsymbol{C} \setminus \{0\})$,
$$\ell_\infty(X, Y, \Phi, \xi, w) \subset \ell_\infty(X, Y, \Phi, \mu, w)$$
if and only if
$$\lim\inf_x \left|\frac{\xi(x)}{\mu(x)}\right|^{w(x)} > 0.$$

Proof:

For the sufficiency of the condition, suppose that
$$\lim\inf_x \left|\frac{\xi(x)}{\mu(x)}\right|^{w(x)} > 0.$$

Then there exists $m > 0$ such that
$$m\,|\mu(x)|^{w(x)} < |\xi(x)|^{w(x)}$$
for all but finitely many $x \in X$.

Let $\phi \in \ell_\infty(X, Y, \Phi, \xi, w)$, $r_1 > 0$ be associated with ϕ and $p_U \in D$, so that
$$\sup_{x\in X} \Phi\left(\frac{[p_U(\xi(x)\phi(x))]^{w(x)}}{r_1}\right) < \infty.$$

Let us choose r such that $r_1 < m\,r$.
For such r, using non decreasing property of Φ, we have

$$\Phi\left(\frac{[p_U(\mu(x)\phi(x))]^{w(x)}}{r}\right) = \Phi\left(\frac{[|\mu(x)|\,p_U(\phi(x))]^{w(x)}}{r}\right)$$
$$\leq \Phi\left(\frac{[|\xi(x)|\,p_U(\phi(x))]^{w(x)}}{m\,r}\right)$$
$$\leq \Phi\left(\frac{[p_U(\xi(x)\phi(x))]^{w(x)}}{r_1}\right),$$

and therefore
$$\sup_{x\in X} \Phi\left(\frac{[p_U(\mu(x)\phi(x))]^{w(x)}}{r}\right) < \infty.$$

Since $p_U \in D$ is arbitrary, it shows that $\phi \in \ell_\infty(X, Y, \Phi, \mu, w)$ and hence
$$\ell_\infty(X, Y, \Phi, \xi, w) \subseteq \ell_\infty(X, Y, \Phi, \mu, w).$$

For the necessity of the condition, assume that
$$\ell_\infty(X, Y, \Phi, \xi, w) \subset \ell_\infty(X, Y, \Phi, \mu, w)$$
but
$$\lim\inf_x \left|\frac{\xi(x)}{\mu(x)}\right|^{w(x)} = 0.$$

Then there exists a sequence (x_k) in X of distinct points such that for each $k \geq 1$, we have
$$k|\xi(x_k)|^{w(x_k)} < |\mu(x_k)|^{w(x_k)} \qquad \dots (5)$$

We now choose $y \in Y$ and $p_V \in D$ such that $p_V(y) = 1$ and define $\phi : X \to Y$ by
$$\phi(x) = \begin{cases} (\xi(x_k))^{-1}\, y, & \text{for } x = x_k,\ k \geq 1, \text{ and} \\ \theta, & \text{otherwise.} \end{cases} \qquad \dots (6)$$

Let $r > 0$. Then for each $p_U \in D$, we have
$$\sup_{x\in X} \Phi\left(\frac{[p_U(\xi(x)\phi(x))]^{w(x)}}{r}\right)$$
$$= \sup_{k\geq 1} \Phi\left(\frac{[p_U(\xi(x_k)\phi(x_k))]^{w(x_k)}}{r}\right)$$
$$= \sup_{k\geq 1} \Phi\left(\frac{[p_U(y)]^{w(x_k)}}{r}\right)$$
$$\leq \Phi\left(\frac{A\,[(p_U(y))^{L(w)}]}{r}\right).$$

This clearly shows that $\phi \in \ell_\infty(X, Y, \Phi, \xi, w)$.

But on the other hand, in view of (5) and (6), we have
$$\sup_{x\in X} \Phi\left(\frac{[p_V(\mu(x)\phi(x))]^{w(x)}}{r}\right)$$

$$= \sup_{k\geq 1} \Phi\left(\frac{[p_V(\mu(x_k)\,\phi(x_k))]^{w(x_k)}}{r}\right)$$

$$= \sup_{k\geq 1} \Phi\left(\frac{p_V\left[\frac{\mu(x_k)}{\xi(x_k)}\,y\right]^{w(x_k)}}{r}\right)$$

$$= \sup_{k\geq 1} \Phi\left(\frac{1}{r}\left|\frac{\mu(x_k)}{\xi(x_k)}\right|^{w(x_k)}[p_V(y)]^{w(x_k)}\right)$$

$$\geq \sup_{k\geq 1} \Phi\left(\frac{k}{r}\right) = \infty.$$

This shows that $\phi \notin \ell_\infty(X, Y, \Phi, \mu, w)$, a contradiction. This completes the proof.

Theorem 6: Let $w \in \ell_\infty(X, R^+)$. Then for any $\xi, \mu \in s(X, C \setminus \{0\})$,

$$\ell_\infty(X, Y, \Phi, \mu, w) \subset \ell_\infty(X, Y, \Phi, \xi, w)$$

if and only if

$$\lim\sup_x \left|\frac{\xi(x)}{\mu(x)}\right|^{w(x)} < \infty.$$

Proof:

For the sufficiency of the condition, assume that

$$\lim\sup_x \left|\frac{\xi(x)}{\mu(x)}\right|^{w(x)} < \infty.$$

Then there exists $d > 0$ such that

$$|\xi(x)|^{w(x)} < d\,|\mu(x)|^{w(x)}$$

for all but finitely many $x \in X$.

Let $\phi \in \ell_\infty(X, Y, \Phi, \mu, w)$, $r_1 > 0$ is associated with ϕ and $p_U \in D$. Then

$$\sup_{x\in X} \Phi\left(\frac{[p_U(\mu(x)\,\phi(x))]^{w(x)}}{r_1}\right) < \infty.$$

Let us choose $r > 0$ such that $d\,r_1 \leq r$, then for such r, using non decreasing property of Φ we have

$$\Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{w(x)}}{r}\right) \leq \Phi\left(\frac{[|\xi(x)|\,p_U(\phi(x))]^{w(x)}}{r}\right)$$

$$\leq \Phi\left(\frac{d\,|\mu(x)|^{w(x)}\,[p_U(\phi(x))]^{w(x)}}{r}\right)$$

$$\leq \Phi\left(\frac{[p_U(\mu(x)\,\phi(x))]^{w(x)}}{r_1}\right)$$

and therefore

$$\sup_{x\in X} \Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{w(x)}}{r}\right) < \infty.$$

Since $p_U \in D$ is arbitrary, it shows that $\phi \in \ell_\infty(X, Y, \Phi, \xi, w)$ and hence

$$\ell_\infty(X, Y, \Phi, \mu, w) \subset \ell_\infty(X, Y, \Phi, \xi, w).$$

For the necessity of the condition, assume that

$$\ell_\infty(X, Y, \Phi, \mu, w) \subset \ell_\infty(X, Y, \Phi, \xi, w)$$

but

$$\lim\sup_x \left|\frac{\xi(x)}{\mu(x)}\right|^{w(x)} = \infty.$$

Then we can find a sequence (x_k) of distinct points in X such that for each $k \geq 1$,

$$|\xi(x_k)|^{w(x_k)} > k\,|\mu(x_k)|^{w(x_k)} \qquad \ldots (7)$$

We now choose $y \in Y$ and $p_V \in D$ such that $p_V(y) = 1$ and define

$$\phi : X \to Y$$

by

$$\phi(x) = \begin{cases} (\mu(x_k))^{-1}\,y, & \text{for } x = x_k,\ k \geq 1, \text{ and} \\ \theta, & \text{otherwise.} \end{cases} \qquad \ldots (8)$$

Let $r > 0$. Then for each $p_U \in D$, we have

$$\sup_{x\in X} \Phi\left(\frac{[p_U(\mu(x)\,\phi(x))]^{w(x)}}{r}\right)$$

$$= \sup_{k\geq 1} \Phi\left(\frac{[p_U(\mu(x_k)\,\phi(x_k))]^{w(x_k)}}{r}\right)$$

$$= \sup_{k\geq 1} \Phi\left(\frac{[p_U(y)]^{w(x_k)}}{r}\right)$$

$$\leq \Phi\left(\frac{A\,[(p_U(y))^{M(w)}]}{r}\right).$$

This clearly shows that $\phi \in \ell_\infty(X, Y, \Phi, \mu, w)$.

But in view of (9) and (10), we have

$$\sup_{x\in X} \Phi\left(\frac{[p_V(\xi(x)\,\phi(x))]^{w(x)}}{r}\right)$$

$$= \sup_{k\geq 1} \Phi\left(\frac{[p_V(\xi(x_k)\,(\mu(x_k))^{-1}\,y)]^{w(x_k)}}{r}\right)$$

$$= \sup_{k\geq 1} \Phi\left(\frac{\left|\frac{\xi(x_k)}{\mu(x_k)}\right|^{w(x_k)}[p_V(y)]^{w(x_k)}}{r}\right)$$

$$\geq \sup_{k\geq 1} \Phi\left(\frac{k}{r}\right)$$

$$= \infty,$$

implies that

$$\phi \notin \ell_\infty(X, Y, \Phi, \xi, w).$$

This leads to a contradiction and completes the proof.

When Theorem 5 and Theorem 6 are combined, we get

Theorem 7: If $w \in \ell_\infty(X, R^+)$ and $\xi, \mu \in s(X, C \setminus \{0\})$, then

$$\ell_\infty(X, Y, \Phi, \xi, w) = \ell_\infty(X, Y, \Phi, \mu, w)$$

if and only if

$$0 < \lim\inf_x \left|\frac{\xi(x)}{\mu(x)}\right|^{w(x)} \leq \lim\sup_x \left|\frac{\xi(x)}{\mu(x)}\right|^{w(x)} < \infty.$$

Corollary 8:

Let $w \in \ell_\infty(X, R^+)$ *and* $\xi \in s(X, C \setminus \{0\})$. Then

(i) $\ell_\infty(X, Y, \Phi, \xi, w) \subset \ell_\infty(X, Y, \Phi, w)$

if and only if

$$\lim\inf_x |\xi(x)|^{w(x)} > 0;$$

(ii) $\ell_\infty(X, Y, \Phi, w) \subset \ell_\infty(X, Y, \Phi, \xi, w)$
if and only if
$\lim\sup_x |\xi(x)|^{w(x)} < \infty$; and

(iii) $\ell_\infty(X, Y, \Phi, w) = \ell_\infty(X, Y, \Phi, \xi, w)$
if and only if
$0 < \lim\inf_x |\xi(x)|^{w(x)} \le \lim\sup_x |\xi(x)|^{w(x)} < \infty$.

Proof:
By considering the function μ on X such that

$$\mu(x) = 1 \text{ for all } x \in X$$

in Theorems 5 and 6 and 7, one can easily obtain the assertions (i), (ii) and (iii) respectively.

Theorem 9: If $\xi \in s(X, C \setminus \{0\})$, $w \in \ell_\infty(X, R^+)$ and $\nu : X \to R^+$, then
$\ell_\infty(X, Y, \Phi, \xi, w) \subset \ell_\infty(X, Y, \Phi, \mu, \nu)$
if and only if

(i) $\lim\sup_x \frac{\nu(x)}{w(x)} < \infty$;

and

(ii) $\lim\inf_x \left|\frac{\xi(x)}{\mu(x)}\right|^{w(x)} > 0$.

Proof:
Proof easily follows from Theorem 1 and Theorem 5.

In the forthcoming theorems, we shall deal with the classes $\ell_\infty(X, Y, \Phi, \xi, w)$ to investigate some results that characterize the linear topological structures of $\ell_\infty(X, Y, \Phi, \xi, w, L)$ by endowing it with suitable natural paranorm.

As far as the linear space structure of the class over the field C of complex numbers is concerned, we throughout take pointwise operations i.e.,for functions ϕ, ψ and scalar α,

$$(\phi + \psi)(x) = \phi(x) + \psi(x)$$

and

$$(\alpha\,\phi)(x) = \alpha\,\phi(x),\ x \in X.$$

Theorem 10: $\ell_\infty(X, Y, \Phi, \xi, w, L)$ forms a linear space over the field **C** with respect to the pointwise vector operations.

Proof:
Suppose $\phi, \psi \in \ell_\infty(X, Y, \Phi, \xi, w, L)$, $r_1 > 0$ and $r_2 > 0$ are associated with ϕ and ψ respectively and $\alpha, \beta \in C$. Let $p_U \in D$ be given. Then we have

$$\sup_{x \in X} \Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{w(x)/L}}{r_1}\right) < \infty;$$

and

$$\sup_{x \in X} \Phi\left(\frac{[p_U(\xi(x)\,\psi(x))]^{w(x)/L}}{r_2}\right) < \infty.$$

We now choose r such that

$2\,r_1 A[\alpha] \le r$ and $2\,r_2 A[\beta] \le r$.

For such r, using non decreasing and convex properties of Φ we have

$$\sup_{x \in X} \Phi\left(\frac{[p_U\{\xi(x)\,(\alpha\phi(x) + \beta\psi(x))\}]^{w(x)/L}}{r}\right)$$
$$\le \sup_{x \in X} \Phi\left[\frac{A[\alpha]}{r}[p_U(\xi(x)\,\phi(x))]^{w(x)/L} + \frac{A[\beta]}{r}[p_U(\xi(x)\,\psi(x))]^{w(x)/L}\right]$$
$$\le \frac{1}{2}\sup_{x \in X} \Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{w(x)/L}}{r_1}\right) + \frac{1}{2}\sup_{x \in X} \Phi\left(\frac{[p_U(\xi(x)\,\psi(x))]^{w(x)/L}}{r_2}\right)$$
$$< \infty.$$

This implies that $\alpha\phi + \beta\psi \in \ell_\infty(X, Y, \Phi, \xi, w, L)$ and so $\ell_\infty(X, Y, \Phi, \xi, w, L)$ forms a linear space over C.

Let $\phi \in \ell_\infty(X, Y, \Phi, \xi, w, L)$ and $p_U \in D$, define a function

$$H_U : \ell_\infty(X, Y, \Phi, \xi, w, L) \to R \text{ by}$$

$H_U(\phi)$

$$= \inf\left\{ r > 0 : \sup_{x \in X} \Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{w(x)/L}}{r}\right) \le 1 \right\} \quad (9)$$

We prove below that $\ell_\infty(X, Y, \Phi, \xi, w, L)$ forms a paranormed space with respect to H_U.

Theorem 11 : If $\inf_x w(x) = l > 0$, then $(\ell_\infty(X, Y, \Phi, \xi, w, L), H_U)$ forms a paranormed space.
Proof:

Since $\Phi(0) = 0$ and therefore

$$H_U(\theta) = 0.$$

Also,

$$H_U(-\phi) = H_U(\phi)$$

and so PN_1 and PN_2 are obvious.
For PN_3,
for any $\phi \in \ell_\infty(X, Y, \Phi, \xi, w, L)$, we denote

$$R(\phi) = \left\{ r > 0 : \sup_{x \in X} \Phi\left(\frac{[p_U(\xi(x)\,\phi(x))]^{w(x)/L}}{r}\right) \le 1 \right\}. \quad (10)$$

Now for $\phi, \psi \in \ell_\infty(X, Y, \Phi, \xi, w, L)$, consider

$$r_1 \in R(\phi) \text{ and } r_2 \in R(\psi).$$

Then clearly by the convexity of Φ, we have

$$\sup_{x \in X} \Phi\left(\frac{[p_U\{\xi(x)(\phi(x) + \psi(x))\}]^{w(x)/L}}{r_1 + r_2}\right) \le 1.$$

In view of (10),this shows that

$$r_1 + r_2 \in R(\phi + \psi)$$

and therefore

$$H_U(\phi + \psi) \le r_1 + r_2$$

for each $r_1 \in R(\phi)$ and $r_2 \in R(\psi)$,
which implies that

$$H_U(\phi + \psi) \le H_U(\phi) + H_U(\psi)$$

i.e., PN_3 holds.
Finally we show the continuity of scalar multiplication.
(i) Let (ϕ_n) be a sequence of functions in $\ell_\infty(X, Y, \Phi, \xi, w, L)$ such that

$$H_L(\phi_k) \to 0 \text{ as } k \to \infty$$

and (α_k) a sequence of scalars such that $\alpha_k \to \alpha$.

We show that

$$H_L(\alpha_k \phi_k) \to 0.$$

Now,

$H_L(\alpha_k \phi_k)$

$$= \inf \{ r : \sup_{x \in X} \Phi \left(\frac{[p_L(\xi(x)\, \alpha_k\, \phi_k(x))]^{w(x)/L}}{r} \right) \leq 1 \}$$

$$= \inf \{ r : \sup_{x \in X} \Phi \left(\frac{\|\alpha_k|^{w(x)/L} [p_L(\xi(x)\, \phi_k(x))]^{w(x)/L}}{r} \right) \leq 1 \}$$

$$\leq \inf \{ r : \sup_{x \in X} \Phi \left(\frac{s^{w(x)/L} [p_L(\xi(x)\, \phi_k(x))]^{w(x)/L}}{r} \right) \leq 1 \}$$

where $s = \sup_k |\alpha_k|$.

Since $s^{w(x)/L} \leq s$ if $s \geq 1$ and $s^{w(x)/L} < 1$ for $s < 1$,
so taking $r = t\,A[s]$, then we have

$H_L(\alpha_k \phi_k)$

$$\leq \inf \{ r : \sup_{x \in X} \Phi \left(\frac{A[s]\, [p_L(\xi(x)\, \phi_k(x))]^{w(x)/L}}{r} \right) \leq 1 \}.$$

$$= \inf \{ t\,A[s] : \sup_{x \in X} \Phi \left(\frac{[p_L(\xi(x)\, \phi_k(x))]^{w(x)/L}}{t} \right) \leq 1 \}$$

$= A[s]\, H_L(\phi_k)$

implies that

$$H_L(\alpha_k \phi_k) \to 0 \text{ as } H_L(\phi_k) \to 0 \text{ as } k \to \infty.$$

(ii) Let $\alpha_k \to 0$ as $k \to \infty$ and ϕ be any element in $\ell_\infty(X, Y, \Phi, \xi, w, L)$. Then the proof of

$$H_L(\alpha_k \phi) \to 0 \text{ as } k \to \infty$$

follows analogously as proved in Theorem 3.3 Srivastava and Pahari(2011).

Hence $\ell_\infty(X, Y, \Phi, \xi, w, L)$ forms a paranormed space.This completes the proof.

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